EXAMINATION IN CHAOS

2005-10-21, 14–19

Aids: TEFYMA (or other comparable table) and pocket calculator.

Complete solution required for each task. The examination consists of one theoretical part and one part with problems, each giving 12 points as a maximum. To pass, a reasonable distribution of points is required, apart from a certain minimum score.

THEORETICAL PART.

1. a) Consider a map $x_{n+1} = f(x_n)$. Define the concept 'attractive fixed point'. Show that a fixed point x^* is attractive if $|f'(x^*)| < 1$. (2p)

   b) Consider the dripping faucet ('droppande kranen') in the chaotic region. How is it possible to determine if a 'first return map' exists. Explain briefly the difference between chaotic behaviour and stochastic behaviour! (2p)

2. Consider the complex map $z(j+1) = z(j)^2 + c$, where z and c are complex numbers.

   a) Define the Julia set. (1p)

   b) What does the Julia set look like for $c = 0$. Motivate briefly starting from the definition in a) (1p)

   c) Describe how it is possible to construct the Mandelbrot set on a computer. (1p)

   d) Which is the Julia set of the complex map

   $$z(j+1) = \frac{z(j)^2 + 1}{2z(j)}$$

   The map is obtained when solving the equation $g(z) = z^2 - 1 = 0$ with the Newton-Raphson method. Motivate your answer. (1p)

3. Consider a three-dimensional flow in phase space, (x_1, x_2, x_3).

   a) Define the Lie derivative. (1p)

   b) How does the phase space volume vary in time if the Lie derivative has the value '$-c$', where 'c' is a positive constant. In a complete answer, you are supposed to give an explicit formula defining how the volume varies. (1p)

   c) Assume that the Lie derivative is negative. Is there some more necessary condition which must be fulfilled for a strange attractor to exist? (1p)

   d) Consider now a two-dimensional Poincaré section of the flow. If a strange attractor is formed, which is its box dimension (the Kaplan-Yorke conjecture)? Do not forget to explain the parameters in the formula. (1p)

PROBLEMS

4. A one-dimensional map, $x_{n+1} = f(x_n)$, is defined as

   $$f(x) = \begin{cases} ax & 0 \le x \le \frac{1}{2} \\ a(1-x) & \frac{1}{2} \le x \le 1. \end{cases}$$

   Consider this map for different values of the parameter a, $0 \le a \le 2$.

   a) Determine the fixed points of the map. (1p)

   b) Determine the Lyapunov exponent of the map. For which values of a is the system chaotic? (1p)

c) Determine the second return map for $a = 2$, for example by drawing a figure to illustrate $f^{(2)}(x) = f(f(x))$, $0 \leq x \leq 1$. Note that you can easily check your answer by calculating $f(f(x))$ for a few values of x. (1p)

5. Consider two different variations of the Cantor set:

a) In each iteration step, split each connected interval into four parts and remove the second interval from the left (the two first steps are illustrated in the figure to the left below).

b) Perform a similar iterative process as in a) but in each step, treat the remaining three parts as separate intervals, which are devided into four parts where one is removed etc. (see the figure to the right below where the different parts are slightly replaced vertically to indicate where they are disconnected).

Calculate the fractal dimension of the two sets which are formed in a) and b),respectively, e.g. by using the formula for self-similar fractals. (2p)

c) If L is the length of the original line, determine the number of lines with length $L/16$ needed to cover the two fractals (note that you should consider the fractal formed after an 'infinite' number of iterations). Discuss if the result you get is qualitatively consistent with the calculated dimensions. (1p)

6. Consider the following system of non-linear differential equations:

$$\begin{cases} \dot{x} = r\left(\frac{c}{x} - d\right) + ax - bxy \\ \dot{y} = r\left(b - \frac{a}{y}\right) - cy + dxy \end{cases}$$

where a, b, c, and d are positive constants and r, x and y are scalar variables, $x > 0$, $y > 0$. When $r = 0$, the system can be seen as an interaction between the population of a prey (bytesdjur) and a predator (rovdjur) species.

a) Show that $(x, y) = \left(\frac{c}{d}, \frac{a}{b}\right)$ is a fixed point for all r. (1p)

b) Investigate the stability of the fixed point $(x, y) = \left(\frac{c}{d}, \frac{a}{b}\right)$ for positive and negative values r in the limit that $|r| \to 0$ (i.e. you can neglect higher orders of r compared with lower orders). You may assume that $b^2c > ad^2$. (2p)

7. Consider the map

$$\begin{cases} x(j+1) = 2x(j) + y(j)^{\alpha} \\ y(j+1) = x(j+1) + b\left(x\left(j\right) + y\left(j\right)\right) \end{cases}$$

where x and y are defined modulo 1

a) Rewrite the map in standard form, i.e.

$$\begin{cases} x(j+1) = f_1\left(x\left(j\right), y\left(j\right)\right) \\ y(j+1) = f_2\left(x\left(j\right), y\left(j\right)\right) \end{cases} . \qquad (0.5\text{p})$$

b) Find values with $b \neq 0$ and $\alpha \neq 0$ so that the map becomes area-conserving. (1p)

c) Consider the values of b and α calculated in b). Are the Lyapunov exponents independent of the initial values of an iteration? Motivate! Calculate the Lyapunov exponents for some iteration. (1.5p)