Solutions to selected problems, chapter 2

2.3

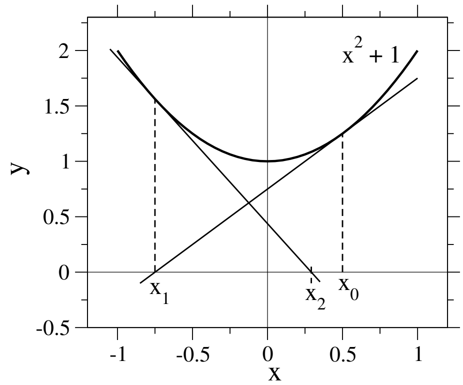
2.3 Consider the function $g(x)$, i.e. $g(x) = x^2+1$. According to the Newton-Raphson method:

$$x_{j+1} = x_j - \frac{g(x_j)}{g'(x_j)} = x_j - \frac{x_j^2+1}{2x_j} = \frac{x_j^2-1}{2x_j} = F(x_j)$$

where we have thus defined $F(x_j)$. If the right side is real, also the left side will be real. Thus, if we start with a real x_0, x_j will be real for all j which means that we will not find any complex roots.

We first try to find out graphically if a two-periodic cycle exists:

Consider the figure to the right. We start with some arbitrary x_0 and determine where the tangent hits $y = 0$ according to the Newton-Raphson method. We thus get the value x_1 and draw the tangent from there, which hits $y = 0$ at x_2. For a two-periodic cycle to exist, x_2 should coincide with x_0 ($x_2 = x_0 = x^*$). From the figure, it appears that this will be the case if x_0 and x_1 are placed symmetrically around the y-axis ($|x_0| = |x_1|$).



Considering the symmetry in the figure, we can also conclude that $|x_0| < |x^*| < |x_1|$. The figure is drawn with $x_0 = 0.5$ which leads to $x_1 = 0.75$. According to the calculations below, $x^* = \pm 1/\sqrt{3} \approx 0.577$ which is in the expected interval.

In order to find the x-value of the two-periodic cycle analytically, we determine the second return map:

$$x_{j+2} = \frac{x_{j+1}^2 - 1}{2x_{j+1}} = \frac{(x_j^2-1)^2 - 4x_j^2}{4x_j(x_j^2-1)} = F^{(2)}(x_j)$$

A two-periodic cycle with a fixed point x^* is a solution of $x^* = F^{(2)}(x^*)$, i.e.

$$4\,(x^*)^2\left[(x^*)^2 - 1\right] = \left[(x^*)^2 - 1\right]^2 - 4\,(x^*)^2$$

With $y = (x^*)^2$, we get a second order equation in y which is straightforward to solve: $y = -1$ or $y = 1/3$. Thus,

$$x^* = \pm i \qquad \text{or} \qquad x^* = \pm\frac{1}{\sqrt{3}}$$

Since we start with a real x_0, the possible cycle corresponds to $x^* = \pm 1/\sqrt{3}$. This solution is also consistent with the discussion using the figure above.

In the calculations above, it has essentially been sufficient to apply the Newton-Raphson method to find the solution. Now we are asked to determine the stability of the two-periodic cycle and then we must use what we learned towards the end of chapter 2, namely that the the two-periodic cycle is stable if the absolute of the derivative of the second return map is smaller than one.

$$F^{(2)}(x) = \frac{(x^2-1)^2 - 4x^2}{4x(x^2-1)} = \frac{x^2-1}{4x} - \frac{x}{x^2-1}$$

and thus

$$\frac{dF^{(2)}(x)}{dx} = \frac{8x^2 - 4(x^2-1)}{16x^2} - \frac{(x^2-1) - 2x^2}{(x^2-1)^2} = \frac{1}{4} + \frac{1}{4x^2} + \frac{x^2+1}{(x^2-1)^2}$$

i.e.

$$\frac{dF^{(2)}(\pm 1/\sqrt{3})}{dx} = \frac{1}{4} + \frac{3}{4} + \quad \text{`something positive'} \quad > 1,$$

i.e. the 2-periodic cycle is unstable.

2.6 With $f(x) = rx(1-x)$ we get

$$f^{(2)}(x) = f(f(x)) = rrx(1-x)(1-rx(1-x))$$

x^* is a fixed point to $f^{(2)}(x)$ if $x^* = f^{(2)}(x^*)$, i.e.

$$x - r^2x(1-x)(1-rx(1-x)) = 0$$

where we have dropped the '*', referring to the fixed point as x. This equation has a solution if either $X = 0$ or

$$1 - r^2(1-x)(1-rx(1-x)) = 0 \Longleftrightarrow p(x) = x^3 - 2x^2 + x(1+\frac{1}{r}) - \frac{1}{r} + \frac{1}{r^2} = 0 \quad (1)$$

where we have introduced the notation $p(x)$ for the 3rd degree polynomial. The equation $p(x) = 0$ with $p(x)$ of 3rd order is very difficult (but possible) to solve in the general case. We will therefore use the property that 'If x is a fixed point to $f(x)$, it is also a fixed point to $f^{(2)}(x)$. The fixed points to f(x) are

$$x = 0 \quad and \quad x = 1 - 1/r$$

We have already noticed that x is a fixed point also of $f^{(2)}(x)$. If $x = 1 - 1/r$ is also a fixed point it means that $p(x)$ can be written as

$$p(x) = (x^2 + \alpha x + \beta)(x - (1 - 1/r)) \quad (2)$$

In principal, there should be a constant in front of the x^2-term but because there is no constant in front of the x^3 term in $p(x)$, we can immediately conclude that this constant must be equal to one. The standard method is now to divide $p(x)$ with $x - (1 - 1/r)$ to find the constants α and β. We find it easier, however, to perform the products in eq. 2 and identify the coefficients when comparing with p(x) in eq. 1. This leads to the equations

$$\begin{cases} \alpha - 1 + 1/r = -2 \\ \beta - \alpha + \alpha/r = 1 + 1/r \\ -1/r + 1/r^3 = -\beta + \beta/r \end{cases}$$

where $\alpha = -(r+1)/r$ is obtained from the first equation and $\beta = (r+1)/r^2$ from the third equation. Furthermore, these values of α and β do also satisfy the second equation.

The two additional fixed points of $f^{(2)}(x)$ are now obtained from

$$x^2 - \frac{r+1}{r}x + \frac{r+1}{r^2} = 0$$

with the solutions,

$$x_{1,2} = \frac{1}{2r}\left(r + 1 \pm \sqrt{(r+1)(r-3)}\right)$$

The roots are real if $r > 3$ (when the fixed point, $x = 1 - 1/r$ becomes unstable).

For $r = 3.2$: $x_1^* = 0.7995$, $x_2^* = 0.5130$.

2.7 The period-doubling occurs when the two fixed points of $f^{(2)}(x)$ which are not simultaneously fixed points of f become unstable. According to exercise 2.6, they are

$$x_{1,2}^* \equiv x_{1,2} = \frac{1}{2r}\left(r + 1 \pm \sqrt{(r+1)(r-3)}\right)$$

They are unstable when $|\frac{d}{dx}(f^{(2)}(x_{1,2}))| = |D(f^{(2)}(x_{1,2}))| > 1$.

With

$$f^{(2)}(x) = r^2\left(x - (r+1)x^2 + 2rx^3 - rx^4\right),$$

the derivative becomes:

$$D(f^{(2)}(x)) = r^2\left(1 - 2(r+1)x + 6rx^2 - 4rx^3\right)$$

We should find the value of this derivative at $x_{1,2}$. Calculate first:

$$x_{1,2}^2 = \frac{1}{4r^2}\left((r+1)^2 \pm 2(r+1)\sqrt{(r+1)(r-3)} + (r+1)(r-3)\right)$$

and

$$x_{1,2}^3 = \frac{1}{8r^3}\left((r+1)^3 \pm 2(r+1)^2\sqrt{(r+1)(r-3)} + (r+1)^2(r-3)\right.$$
$$\left.\pm (r+1)^2\sqrt{(r+1)(r-3)} + 2(r+1)(r-3) \pm (r+1)(r-3)\sqrt{(r+1)(r-3)}\right)$$

which can be simplified somewhat

$$\begin{aligned} x_{1,2}^3 &= \frac{1}{8r^3}\left((r+1)^3 \pm 3(r+1)^2\sqrt{(r+1)(r-3)}\right. \\ &+ \left.3(r+1)^2(r-3) \pm (r+1)(r-3)\sqrt{(r+1)(r-3)}\right) \end{aligned}$$

Now insert $x_{1,2}$ into $D(f^{(2)}(x))$:

$$\begin{aligned} \frac{1}{r^2}D(f^{(2)}(x_{1,2})) &= 1 - \frac{r+1}{r}\left(r + 1 \pm \sqrt{(r+1)(r-3)}\right) \\ &+ \frac{3}{2r}\left((r+1)^2 \pm 2(r+1)\sqrt{(r+1)(r-3)} + (r+1)(r+3)\right) \\ &- \frac{1}{2r^2}\left((r+1)^3 \pm 3(r+1)^2\sqrt{(r+1)(r-3)} + \right. \\ &+ \left.3(r+1)^2(r-3) \pm (r+1)(r-3)\sqrt{(r+1)(r-3)}\right) \end{aligned}$$

Explicit calculations show that the terms containing the $\sqrt{(r+1)(r-3)}$-factor cancel.

These terms must cancel because of the ±-signs because it is a general property that $D(f^{(2)}(x_1)) = D(f^{(2)}(x_2))$.

The derivative is now obtained as:

$$\begin{aligned} D(f^{(2)}(x_{1,2})) &= r^2 - r(r+1)^2 + \frac{3r}{2}(r+1)^2 + \frac{3r}{2}(r+1)(r-3) \\ &+ \frac{1}{2}(r+1)^3 - \frac{3r}{2}(r+1)^2(r-3) = \ldots \\ &= -r^2 + 2r + 4 \end{aligned}$$

The fixed points $x_{1,2}$ are stable if:

$$\left|D(f^{(2)}(x_{1,2})\right| < 1 \Leftrightarrow |r^2 - 2r - 4| < 1 \Leftrightarrow \left\{ \begin{array}{ll} r^2 - 2r - 4 < 1 & \text{and} \\ r^2 - 2r - 4 > -1 & \end{array} \right.$$

In order to get a better understanding about when these inequalities are fulfilled, it is adviceable to draw the function $r^2 - 2r - 4$. Such a graph should also be helpful to understand the analytical calculations below.

The first equation is fulfilled *if*:

$$(r-1)^2 - 6 < 0 \Leftrightarrow \left\{ \begin{array}{ll} r - 1 < \sqrt{6} & \text{and} \\ r - 1 > -\sqrt{6} & \end{array} \right. \Leftrightarrow \left\{ \begin{array}{ll} r < 1 + \sqrt{6} & \text{and} \\ r > 1 - \sqrt{6} & \end{array} \right.$$

The second equation is fulfilled *if*:

$$(r-1)^2 - 4 > 0 \Leftrightarrow \left\{ \begin{array}{ll} r - 1 < -2 & \text{and} \\ r - 1 > 2 & \end{array} \right. \Leftrightarrow \left\{ \begin{array}{ll} r < -1 & \text{and} \\ r > 3 & \end{array} \right.$$

Since $r > 0$, the condition for stability of $x^*_{1,2}$ is

$$3 < r < 1 + \sqrt{6}$$

Thus, the fixed points $x^*_{1,2}$ become unstable at $r = 1 + \sqrt{6} \approx 3.4495$, i.e. the bifurcation from 2- to 4-periodicity takes place at this r-value.

2.11 The Lyapunov exponent for an iteration $x_{j+1} = f(x_j)$ is defined as

$$\lambda = \lim_{N \to \infty} \left[\frac{1}{N} \sum_{j=1}^{N} \ln \mid f'(x_j) \mid \right]$$

λ is thus the mean value of f' at the points x_j passed by the iteration (in the limit $j \to \infty$). Thus, if the iteration converges to a fixed point x^*, x_j will in practice become equal to x^* for all $j's$ which are 'large enough'. Consequently, for large enough N, the average will be completely dominated by $\ln \mid f'(x^*) \mid$ and $\lambda = \ln \mid f'(x^*) \mid$. As the fixed point is stable, $\mid f'(x^*) \mid < 1$, i.e. $\lambda < 0$.

If $\{x^*_1, x^*_2, \ldots x^*_n\}$ is an n-periodic attractor, an iteration will pass through these points and come back to the initial value after n steps. Consequently, the Lyapunov exponent will result from the average of these values, i.e.

$$\lambda = \frac{1}{n} \sum_{k=1}^{n} \ln \mid f'(x^*_k) \mid$$

This expression can be rewritten using the derivative of $f^{(n)}$. Thus, the property, $\ln(ab) = \ln(a) + \ln(b)$, is first applied:

$$\frac{1}{n}\sum_{k=1}^{n} \ln \mid f'(x_k^*) \mid = \frac{1}{n} \ln \prod_{k=1}^{n} \mid f'(x_k^*) \mid$$

We can now use Eq. (2.15) which takes the form (x_i^* is one value on the n-periodic cycle):

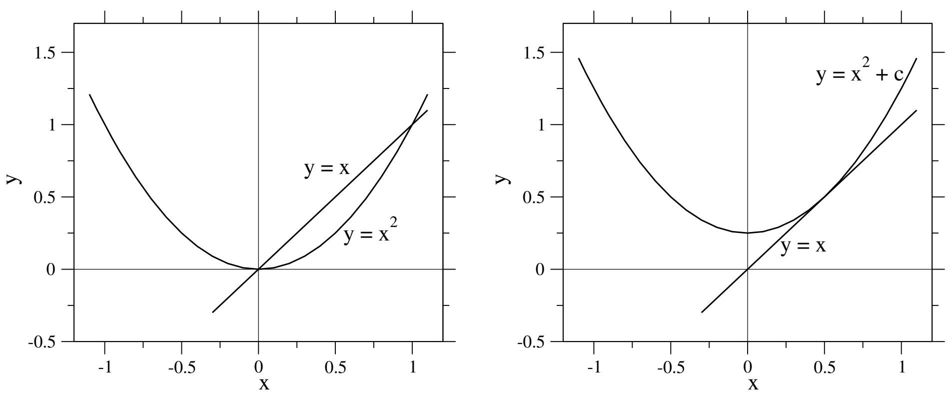
$$Df^{(n)}(x_i^*) = \prod_{k=1}^{n} f'(x_k^*)$$

and thus

$$\lambda = \frac{1}{n} \ln \mid Df^{(n)}(x_i^*) \mid$$

Because x_k^* is a stable fixed point to $f^{(n)}$, $\mid Df^{(n)}(x_i^*) \mid < 1$, and thus $\lambda < 0$ also in this case. Note also that the equations above show that $f^{(n)}$ has the same derivative at all its fixed points which are part of the cycle.

2.14 From the figures below, it is evident that the tangent bifurcation occurs when the two solutions of the equation $f(x) = x$ ($f(x) = x^2 + c$) become equal



Let us first ignore the mod-operation:

$$x^2 + c = x \quad \Rightarrow \quad x = \frac{1}{2} \pm \sqrt{\frac{1}{4} - c}$$

where the two solutions become equal for $c = 1/4$. With the mod-operation, the general solution is thus $c = 1/4 + m$, where m is an integer.