Solution problem 4.7

4.7 We consider a simplified Nicholson-Bailey model:

$$\begin{cases} x_1(j+1) &= Rx_1(j)\left(1 - ax_2(j)\right) &= f_1\left(x_1(j), x_2(j)\right) \\ x_2(j+1) &= ax_1(j)x_2(j) &= f_2\left(x_1(j), x_2(j)\right) \end{cases}$$

The Jacobian is thus obtained as:

$$\mathbf{Df}(\mathbf{x}) = \begin{pmatrix} R(1-ax_2) & -Rax_1 \\ ax_2 & ax_1 \end{pmatrix}$$

The fixed point are calculated from the system of equations:

$$\begin{cases} x_1 &= f_1\left(x_1, x_2\right) &= Rx_1\left(1 - ax_2\right) \\ x_2 &= f_2\left(x_1, x_2\right) &= ax_1x_2 \end{cases}$$

with the trivial solution $x_1 = x_2 = 0$ and with the second solution obtained from the equation:

$$\begin{cases} 1 &= R\left(1 - ax_2\right) \\ 1 &= ax_1 \end{cases} \qquad \Longrightarrow \qquad \begin{cases} x_1 &= x_1^* &= 1/a \\ x_2 &= x_2^* &= (R-1)/aR \end{cases}$$

The model is defined only for R and a positive. We note that $x_2^* > 0$ requires $R > 1$.

The stability is determined from the Jacobian at $x_1 = x_1^*$ and $x_2 = x_2^*$.

$$\mathbf{Df}(x_1^*, x_2^*) = \begin{pmatrix} R\left(1 - a\dfrac{R-1}{aR}\right) & -Ra\dfrac{1}{a} \\ a\dfrac{R-1}{aR} & a\dfrac{1}{a} \end{pmatrix} = \begin{pmatrix} 1 & -R \\ \dfrac{R-1}{R} & 1 \end{pmatrix}$$

The eigenvalues $\tilde{h}_1$ and $\tilde{h}_2$ are obtained from

$$\begin{vmatrix} 1-\tilde{h} & -R \\ \dfrac{R-1}{R} & 1-\tilde{h} \end{vmatrix} = 0$$

and thus

$$\tilde{h}_{1,2} = 1 \pm \sqrt{1-R} = 1 \pm i\sqrt{R-1}$$

Complex eigenvalues with absolute values larger than 1 ($|\tilde{h}_{1,2}| > 1$), which means that the trajectory is spiraling outwards (you need not determine if it spirals clockwise or counter clockwise). The equilibrium is unstable.