Solutions to selected problems, chapter 6

6.3 a) Hénon-Heiles potential:

$$V(x,y) = \frac{1}{2}(x^2 + y^2 + 2x^2y - \frac{2}{3}y^3)$$

Polar coordinates:

$$\begin{cases} x = r\cos\theta = r\frac{1}{2}(e^{i\theta} + e^{-i\theta}) \\ y = r\sin\theta = r\frac{1}{2i}(e^{i\theta} - e^{-i\theta}) \end{cases}$$

where we have used Euler's formula ($e^{ix} = \cos x + i\sin x$) which makes is easy to rewrite trigonometric expressions. The third order terms in V are written in polar coordinates,

$$2x^2y - \frac{2}{3}y^3 = 2r^3\cos^2\theta\sin\theta - \frac{2}{3}r^3\sin^3\theta$$

These expressions are rewritten using Euler's formula:

$$\sin^2\theta = -\frac{1}{4}(e^{2i\theta} - 2 + e^{-2i\theta})$$

$$\begin{aligned} \sin^3\theta &= -\frac{1}{4}(e^{2i\theta} - 2 + e^{-2i\theta})\frac{1}{2i}(e^{i\theta} - e^{-i\theta}) \\ &= -\frac{1}{8i}(e^{3i\theta} - 2e^{i\theta} + e^{-i\theta} - e^{i\theta} + 2e^{-i\theta} - e^{-3i\theta}) \\ &= -\frac{1}{4}\left(\frac{1}{2i}(e^{3i\theta} - e^{-3i\theta}) - \frac{3}{2i}(e^{i\theta} - e^{-i\theta})\right) = -\frac{1}{4}\sin 3\theta + \frac{3}{4}\sin\theta \end{aligned}$$

Alternatively, $\sin^3\theta$ can be looked up in a table.
The $\cos^2\theta\sin\theta$ term can be rewritten in a similar way but it is easier to use the derivation above:

$$\cos^2\theta\sin\theta = (1 - \sin^2\theta)\sin\theta = \sin\theta - \sin^3\theta = \frac{1}{4}\sin\theta + \frac{1}{4}\sin 3\theta$$

Consequently

$$2r^3\cos^2\theta\sin\theta - \frac{2}{3}r^3\sin^3\theta = r^3\frac{2}{3}\sin 3\theta$$

and

$$V(x,y) = \frac{1}{2}(r^2 + \frac{2}{3}r^3\sin 3\theta)$$

b) Solved in the compendium.

6.7

$$\begin{cases} x_{n+1} = (\cos\alpha)x_n - (\sin\alpha)(y_n - x_n^2) = f_1(x_n, y_n) \\ y_{n+1} = (\sin\alpha)x_n + (\cos\alpha)(y_n - x_n^2) = f_2(x_n, y_n) \end{cases}$$

Calculate the Jacobian:

$$\mathbf{Df} = \begin{pmatrix} \partial f_1/\partial x_n & \partial f_1/\partial y_n \\ \partial f_2/\partial x_n & \partial f_2/\partial y_n \end{pmatrix} = \begin{pmatrix} \cos\alpha + 2x_n \sin\alpha & -\sin\alpha \\ \sin\alpha - 2x_n\cos\alpha & \cos\alpha \end{pmatrix}$$

Its determinant is

$$\det\{\mathbf{Df}\} = \cos\alpha(\cos\alpha + 2x_n\sin\alpha) + \sin\alpha(\sin\alpha - 2x_n\cos\alpha) = \cos^2\alpha + \sin^2\alpha = 1,$$

which shows that the map is area preserving, because the areas in iteration n and $(n+1)$ are related through

$$\Delta A_{n+1} = |\det\{\mathbf{Df}\}|\Delta A_n$$

6.9 a)

$$\begin{cases} x_{n+1} = y_n \\ y_{n+1} = -\varepsilon x_n + \mu - y_n^2 \end{cases}$$

The Jacobian:

$$\mathbf{Df} = \begin{pmatrix} 0 & 1 \\ -\varepsilon & -2y_n \end{pmatrix}$$

In the present case, $\det\{\mathbf{Df}\} = \varepsilon$, and the area is preserved if

$$|\det\{\mathbf{Df}\}| = |\varepsilon| = 1$$

b) With $\varepsilon = 0$, we get

$$y(j+1) = \mu - y(j)^2$$

which means that the iteration in y is independent of x so the other equation, $x(j+1) = y(j)$ does not really add anything. The iteration in y is equivalent to the logistic map because the logistic map can be rewritten as

$$x_{n+1} = rx_n(1-x_n) = -r\left(x_n^2 - x_n\right) = -r\left(x_n - \frac{1}{2}\right)^2 + \frac{r}{4} = -\frac{1}{r}\left(rx_n - \frac{r}{2}\right)^2 + \frac{r}{4}$$

Thus

$$rx_{n+1} = -\left(rx_n - \frac{r}{2}\right)^2 + \frac{r^2}{4}$$

or

$$rx_{n+1} - \frac{r}{2} = -\left(rx_n - \frac{r}{2}\right)^2 + \frac{r^2}{4} - \frac{r}{2}$$

i.e., we put $y = rx - \frac{r}{2}$ and $\mu = \frac{r^2}{4} - \frac{r}{2}$ Thus when r varies in the interval [0,4], μ varies from 0 to $-1/4$ and then increases to 2. We will thus get a bifurcation diagram which is essentially equivalent to that of the logistic map where for example the interesting region $r = 3 - 4$ corresponds to $\mu = 0.75 - 2.00$.