A Brief Overview on Complex Functions

Andreas Wacker[1]
Mathematical Physics, Lund University
March 3, 2015

1 Complex functions

Complex numbers z can be viewed as the composition of two real numbers (x, y), called the real part $x = \mathrm{Re}\{z\}$ and the imaginary part $y = \mathrm{Im}\{z\}$, with two operations mapping arbitrary complex numbers z_1, z_2 to a result z'.

$$\text{Addition} \quad z' = z_1 + z_2 \text{ where } x' = x_1 + x_2 \text{ and } y' = y_1 + y_2 \tag{1}$$
$$\text{Multiplication} \quad z' = z_1 z_2 \text{ where } x' = x_1 x_2 - y_1 y_2 \text{ and } y' = x_1 y_2 + y_1 x_2 \tag{2}$$

In practice one writes $z = x + \mathrm{i}y$ and applies the common rules for addition and multiplication together with $\mathrm{i}^2 = -1$. We define the

$$\text{conjugation } z^* = \mathrm{Re}\{z\} - \mathrm{i}\mathrm{Im}\{z\}$$
$$\text{absolute value } |z| = \sqrt{\mathrm{Re}\{z\}^2 + \mathrm{Im}\{z\}^2} = \sqrt{zz^*}$$

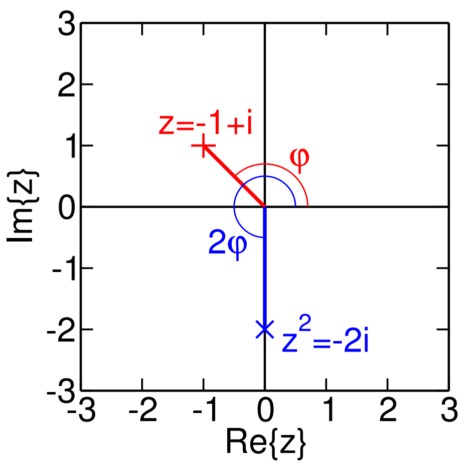

Figure 1: Plane of complex numbers

For illustrative purpose, complex numbers z are displayed in a plane spanned by the real and imaginary axis, see Fig. 1. They can be parameterized by $|z|$, which marks the distance from the origin, and the angle $\varphi \in \mathbb{R}$ with the real axis:

$$z = |z|(\cos\varphi + \mathrm{i}\sin\varphi) = |z|\mathrm{e}^{\mathrm{i}\varphi}$$

The latter relation applies Euler's formula, which will be proven in Eq. (6).

A complex function $z' = f(z)$ has the complex numbers both as arguments z and as function values z'. For arbitrary complex numbers $\alpha = |\alpha|\mathrm{e}^{\mathrm{i}\phi}$, we identify elementary examples for complex functions $z' = f(z)$ as mappings in the complex plane which are defined on the basis of addition and multiplication of complex numbers:

$$f(z) = z + \alpha \text{ (a translation in the plane)} \tag{3}$$
$$f(z) = \alpha z \text{ (rotation by } \phi \text{ and stretching by } |\alpha|) \tag{4}$$
$$f(z) = z^n \text{ (n-fold angle and expansion to } |z'| = |z|^n) \tag{5}$$

Combining these functions, we obtain power series $f(z) = \sum_n a_n z^n$ with arbitrary complex coefficients a_n. Thus, for any real function, which is described by a Taylor series, we can define

[1] Andreas.Wacker@fysik.lu.se

a more general complex function via the corresponding power series. E.g., we have the complex exponential function

$$\mathrm{e}^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \,.$$

For the argument $z = \mathrm{i}\varphi$, where $\varphi \in \mathbb{R}$, we thus find

$$\text{Euler's formula:} \qquad \mathrm{e}^{\mathrm{i}\varphi} = \sum_{j=0}^{\infty} (-1)^j \frac{\varphi^{2j}}{(2j)!} + \mathrm{i} \sum_{j} (-1)^j \frac{\varphi^{2j+1}}{(2j+1)!} = \cos(\varphi) + \mathrm{i}\sin(\varphi) \tag{6}$$

where we used the Taylor expansions of the trigonometric functions. Now it can be shown (see any textbook on complex functions), that

> For any power series $\sum_n a_n z^n$ there is a specific convergence radius R, so that the power series converges for all $|z| < R$ and diverges for $|z| > R$.

In particular, the convergence radius is infinite for the Taylor series of e^z, $\sin(z)$, and $\cos(z)$.

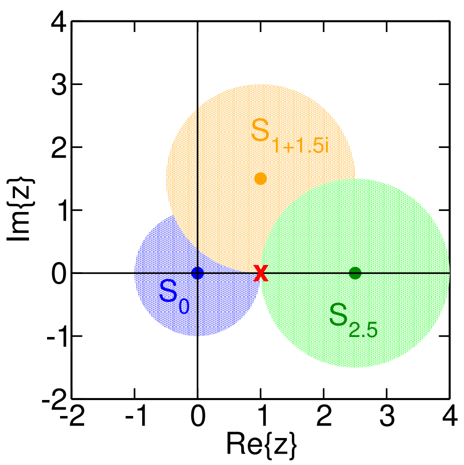

Figure 2: Convergence radius for the Taylor series S_{z_0} of the function $\frac{1}{z-1}$ for different z_0

In contrast the series

$$S_0 = -\sum_{n=0}^{\infty} z^n$$

has the convergence radius $R = 1$. This is related to the fact, that it is the Taylor expansion around $z_0 = 0$ of the function $f(z) = \frac{1}{z-1}$, which has a pole at $z = 1$. However, one may do a Taylor expansion of $f(z)$ around any other point $z_0 \neq 1$ as well, resulting in

$$f(z) = S_{z_0} = -\sum_{n=0}^{\infty} \frac{1}{(1-z_0)^n}(z-z_0)^n$$

This power series has a convergence radius $R = |1 - z_0|$ around z_0. Again, this is just the distance to the pole at $z = 1$. We conclude, that for each point $z \neq 1$, we may choose an appropriate value z_0 so that the Taylor expansion S_{z_0} for $f(z) = \frac{1}{z-1}$ converges in a finite range around z.

2 Complex derivative

Taking the derivative of a complex function is far more intricate than one would think. Its definition is, analogously to the common real functions:

$$f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$$

However, as δz has a real and imaginary component, there are different possible directions of δz in the complex plane. Then the derivative only makes sense, if $f'(z)$ does not depend on the direction chosen. Looking at the fundamental directions with $z = x + \mathrm{i}y$ we find

$$f'(z) = \begin{cases} \lim_{\delta x \to 0} \frac{f(x+\delta x, y) - f(x,y)}{\delta x} & = & \frac{\partial f(x,y)}{\partial x} \\ \lim_{\delta y \to 0} \frac{f(x, y+\delta y) - f(x,y)}{\mathrm{i}\delta y} & = & -\mathrm{i}\frac{\partial f(x,y)}{\partial y} \end{cases}$$

It is straightforward to see, that the elementary functions (3,4,5) satisfy $\mathrm{i}\frac{\partial f(x,y)}{\partial x} = \frac{\partial f(x,y)}{\partial y}$ and consequently the complex derivative $f'(z)$ exists for all power series. One can show:

A complex function $f(z)$ can be written as a power series, which converges within a region around z_0.	$\Leftrightarrow$	The complex derivative $f'(z)$, as well as any higher order derivatives $f^n(z)$ exist for all z in a region around z_0.
These functions are called *analytic* or *holomorphic*, respectively, at z_0.		

Note that the function $f(z) = z^*$ is not holomorphic as $\partial f/\partial x = 1$ and $\partial f/\partial y = -\mathrm{i}$, so that the complex derivative does not exist at any point. The same holds for $f(z) = |z|$.

3 Contour integrals

The integral $\int_{\text{curve}} \mathrm{d}z\, f(z)$ of a complex function along a curve is analogously to a line integral in vector analysis. It is evaluated by parameterizing the curve as $z(t)$, where the real parameter t runs in the interval $t_i < t < t_e$. This provides (which can be seen as a definition of the complex integral)

$$\int_{\text{curve}} \mathrm{d}z\, f(z) = \int_{t_i}^{t_e} \mathrm{d}t\, \frac{\mathrm{d}z(t)}{\mathrm{d}t} f(z(t))$$

Integrals over a closed curve (called contour) $\mathcal{C}$ are of particular relevance. Consider, e.g, the function $f(z) = z^n$ with integer n where $\mathcal{C}$ is a circle going counterclockwise around the origin with radius R. We find with $z(t) = R\mathrm{e}^{\mathrm{i}t}$

$$\int_{\mathcal{C}} \mathrm{d}z\, z^n = \int_0^{2\pi} \mathrm{d}t\, \mathrm{i}R\mathrm{e}^{\mathrm{i}t}\, R^n \mathrm{e}^{n\mathrm{i}t} = \begin{cases} 0 & \text{for } n \neq -1 \\ 2\pi\mathrm{i} & \text{for } n = -1 \end{cases}$$

This example suggests that singularities of the form $1/(z - z_j)$ are related to non-zero contour integrals for contours around z_i. In order to quantify this, we call $R(z_j)$ the *residue* for the function $f(z)$ at the position z_j if

$$f(z) \approx \frac{R(z_i)}{z - z_j} \text{ for } z \approx z_j$$

These residues enter a central theorem for contour integration, as proven in any textbook on complex functions:

> *Residue theorem:* Consider a closed contour $\mathcal{C}$ in the complex plane, which does not cross itself and has a counterclockwise orientation. The complex function $f(z)$ is holomorphic on the contour and in the area inside $\mathcal{C}$ expect for a set of distinct points z_i. Then the contour integral can be evaluated as
>
> $$\int_{\mathcal{C}} \mathrm{d}z\, f(z) = 2\pi\mathrm{i} \sum_{j \text{ with } z_i \text{ inside } \mathcal{C}} R(z_j) \tag{7}$$

Note that singularities like

$$f(z) \approx \frac{A}{(z - z_j)^n} \text{ for } z \approx z_j \text{ with } n \geq 2$$

have $R(z_i) = 0$ and do not contribute to the integral. As can be seen from the example given above, only singularities $\propto 1/(z - z_j)$ matter!

If the function $f(z)$ is holomorphic on and within $\mathcal{C}$, we find in particular

$$\text{Stokes Theorem} \quad \int_{\mathcal{C}} \mathrm{d}z\, f(z) = 0 \tag{8}$$

$$\text{Cauchy's Integral Formula} \quad \int_{\mathcal{C}} \mathrm{d}z\, \frac{f(z)}{z-z_0} = 2\pi \mathrm{i} f(z_0) \tag{9}$$

which are typically proven individually before the residue theorem in textbooks.

4 Applying the residue theorem

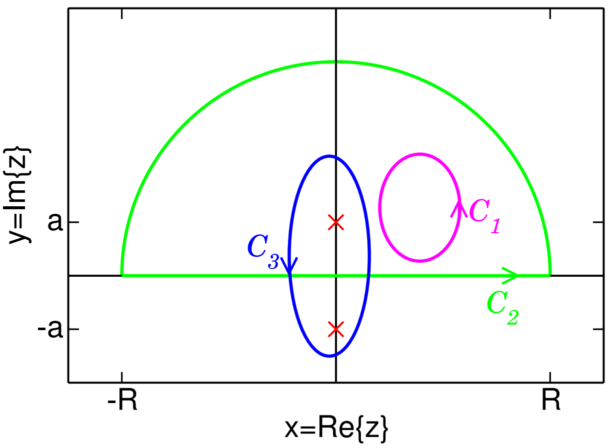

Figure 3: Sketch of different contours

Let us, as an example, consider the function

$$f(z) = \frac{1}{z^2+a^2} = \frac{1}{(z+\mathrm{i}a)(z-\mathrm{i}a)} \text{ for } a \in \mathbb{R}$$

which has poles at $z_1 = \mathrm{i}a$ and $z_2 = -\mathrm{i}a$ with the corresponding residues $R(z_1) = (2\mathrm{i}a)^{-1}$ and $R(z_2) = (-2\mathrm{i}a)^{-1}$. For the contours shown in Fig. 3, we find immediately

$$\int_{\mathcal{C}_1} \mathrm{d}z\, f(z) = 0 \text{ (no pole in } \mathcal{C}_1)$$
$$\int_{\mathcal{C}_2} \mathrm{d}z\, f(z) = \frac{\pi}{a} \text{ (pole at } z_1 = \mathrm{i}a)$$
$$\int_{\mathcal{C}_3} \mathrm{d}z\, f(z) = 0 \text{ (both residues add to zero)}$$

Of particular interest is the contour $\mathcal{C}_2$ in the limit $R \to \infty$. Parameterizing the straight line from $z = -R$ to $z = R$ by $z = x$ with $x \in [-R, R]$ and the half circle by $z = Re^{\mathrm{i}\varphi}$ with $\varphi \in [0, \pi]$, we find

$$\frac{\pi}{a} = \int_{\mathcal{C}_2} \mathrm{d}z\, f(z) = \int_{-R}^{R} \mathrm{d}x\, \frac{1}{x^2+a^2} + \underbrace{\int_0^{\pi} \mathrm{d}\varphi\, \frac{\mathrm{i}Re^{\mathrm{i}\varphi}}{R^2 e^{2\mathrm{i}\varphi}+a^2}}_{\to 0 \text{ for } R\to\infty}$$

This provides us with the integral

$$\int_{-\infty}^{\infty} \mathrm{d}x\, \frac{1}{x^2+a^2} = \frac{\pi}{a}$$

While this result could have been obtained by direct integration using $\frac{\mathrm{d}\arctan(x/a)}{\mathrm{d}x} = \frac{a}{x^2+a^2}$, the method can be also used for more difficult functions, where no antiderivative exists. Such an example is evaluated in great detail at `www.youtube.com/watch?v=MRLa5bk3_R4`.