A physicists point of view on delta-functions

Andreas Wacker[1]
Mathematical Physics, Lund University
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1 Physics slang

The delta-function $\delta(x-x_0)$ is a function, which is zero for $x \neq x_0$ and infinity at $x = x_0$. Most importantly, it satisfies $\int_a^b \mathrm{d}x\, \delta(x-x_0) = 1$ if $a < x_0 < b$. A little bit more general, it can be defined by the property, that for any well-behaved (more precisely continuous) function $f(x)$ we have

$$\int_a^b \mathrm{d}x\, \delta(x-x_0) f(x) = f(x_0) \quad \text{if } a < x_0 < b \tag{1}$$

This characterizes the delta-function very well in practice and most physicists use such a point of view. In this context it is important to realize that one can only evaluate something if there is an integral over the argument of the delta-function, and we also see, that the delta-function cancels the dimension of the integral. Thus remember

- The dimension of the delta-function is the inverse of its argument.
- The delta-function in an expression only makes sense if an integral is available for its liquidation.

Example: You want to describe a standard football situation. The ball is coming from the corner-point at $t_c = 5399$ s with momentum $\mathbf{p}_0$ (air friction neglected), hits Zlatan Ibrahimović's head (or your personal favorite forward) at $t_0 = 5400$ s, and continues into the goal. As the contact time between head and ball is very short while the force is violent, you may describe the force on the ball by

$$\mathbf{F}(t) = \mathbf{U}\delta(t-t_0)\,. \tag{2}$$

As force is in Newton (N) and time in seconds (s), the dimension of $\mathbf{U}$ must be Ns, which is a momentum. Now we calculate the momentum of the ball for $t > t_0$. According to Newton we have $\dot{\mathbf{p}} = \mathbf{F}(t)$ and thus

$$\mathbf{p}(t) = \mathbf{p}_0 + \int_{t_c}^{t} \mathrm{d}t'\, \mathbf{F}(t') = \mathbf{p}_0 + \mathbf{U} \quad \text{for } t > t_0$$

We see, that we obtain a meaningful result after performing an integral over the argument t of the delta function.

While this was simple, we now want to evaluate the energy $E(t)$ of the ball for $t > t_0$. We know from mechanics, that the power transferred to a body moving with velocity $\mathbf{v}(t)$ is $P(t) = \mathbf{v}(t) \cdot \mathbf{F}(t)$ and thus we are tempted to write

$$E(t) = E_0 + \int_{t_c}^{t} \mathrm{d}t'\, \mathbf{v}(t) \cdot \mathbf{F}(t') \stackrel{?}{=} E_0 + \mathbf{v}(t_0) \cdot \mathbf{U} \quad \text{for } t > t_0$$

[1] Andreas.Wacker@fysik.lu.se

However this is problematic, as the velocity is not defined at t_0, where it actually changes between $\mathbf{p}_0/m$ and $(\mathbf{p}_0 + \mathbf{U})/m$. With other words $\mathbf{v}(t)$ is not continuous at t_0 and thus the use of the delta-function does not make sense in this context.

This intricacy becomes more obvious if we resolve the process in time. In reality there is a finite force (here assumed to be constant for simplicity) for a finite duration $\epsilon \sim 1$ ms in time:

$$\mathbf{F}_\epsilon(t) = \begin{cases} \frac{\mathbf{U}}{\epsilon} & \text{for } t_0 - \epsilon/2 < t < t_0 + \epsilon/2 \\ 0 & \text{otherwise} \end{cases}$$

Using $\mathbf{F}_\epsilon(t)$, we can obviously evaluate $p(t)$ and $E(t)$ without any conceptional problems. However, it is sometimes much easier to work with $\mathbf{F}(t)$ from Eq. (2), in particular if the precise time-dependence of the force is not known. This is possible if all other functions within the integral have a negligible time-dependence on the time scale ϵ (such as the position $\mathbf{r}(t)$ of the ball or the camera resolution, e.g.).

From this example we conclude:

The delta function $\delta(x - x_0)$ describes a process which only occurs within a small range Δx around x_0. A more detailed resolution is not required, as long as all other functions in the integral are approximately constant on the scale Δx.

2 Mathematics: Distributions

The delta-function is not a function in a mathematical sense. It can however be defined as a limiting case for functions $\delta_\epsilon(x)$, which are well-behaving for all $\epsilon > 0$, and satisfy

$$\lim_{\epsilon\to 0} \delta_\epsilon(x) = 0 \text{ for } x \neq 0 \quad \text{and} \quad \int_{-\infty}^{\infty} \mathrm{d}x\, \delta_\epsilon(x) = 1$$

Examples for such functions are

$$\delta_\epsilon(x) = \frac{1}{2\pi} \frac{\epsilon}{x^2 + \epsilon^2/4} \quad \text{or} \quad \delta_\epsilon(x) = \frac{1}{\epsilon\sqrt{\pi}} \exp\left(-\frac{x^2}{\epsilon^2}\right)$$

Note that in both cases the function $\delta_\epsilon(x)$ has the dimension of $1/x$. Then the integral over an expression containing the delta-function (1) can be defined as

$$\int_a^b \mathrm{d}x\, \delta(x - x_0) f(x) = \lim_{\epsilon\to 0} \int_a^b \mathrm{d}x\, \delta_\epsilon(x - x_0) f(x)$$

which provides the previous result $f(x_0)$, provided that $f(x)$ is well behaved in the region around x_0. The formal theory behind this is known as distribution theory. In practice this procedure is helpful for analyzing situations, where the meaning of the delta-function is not apriori clear, as in the example for the energy transfer to the football discussed above.

3 Delta function with a more complicated argument

Quite frequently one has to evaluate expressions like $\int_a^b \mathrm{d}x\, \delta(g(x)) f(x)$, where $g(x)$ is an arbitrary function. Obviously, this picks the points $x_i \in [a, b]$, where $g(x_i) = 0$. The first idea is to write

$$\int_a^b \mathrm{d}x\, \delta(g(x)) f(x) = \sum_{x_i} f(x_i) \quad \textbf{wrong!}$$

That this expression must be wrong can be seen by considering the dimensions. As the dimension of the delta -function is its inverse argument, the left-hand side has the dimension $[xf/g]$. In contrast, the right-hand side has the dimension $[f]$. This discrepancy in dimension, can be cured by dividing by the derivative $g'(x)$. The correct expression is

$$\int_a^b \mathrm{d}x\, \delta(g(x))f(x) = \sum_{x_i} \frac{f(x_i)}{|g'(x_i)|} \quad \text{where the } x_i \text{ satisfy } g(x_i) = 0 \text{ with } a < x_i < b \tag{3}$$

Note the absolute value for the derivative of $g(x)$ in the denominator! In case the derivative $g'(x_i)$ becomes zero, the delta-function is undefined and one has to examine the underlying physical situation carefully.

Example:

$$\int_{-\infty}^{\infty} \mathrm{d}x\, \delta(\alpha(x^2 - a^2)) \sin(kx) = \frac{\sin(ka)}{2\alpha a} + \frac{\sin(-ka)}{2\alpha a}$$

The proof of Eq. (3) is straightforward. At first divide the integration-interval into parts, where $g(x)$ is monotonously increasing or decreasing. In each interval, substitute $y = g(x)$, and use $\mathrm{d}x = \mathrm{d}y/g'(x)$. Then Eq. (1) provides the desired result for monotonously increasing parts with $g'(x) > 0$. For monotonously decreasing parts, $g'(x) < 0$, the upper bound in y is smaller than the lower bound. However, Eq. (1) requires the upper bound to be larger. Exchanging bounds provides a minus sign, which is taken into account by the absolute value of $g'(x)$ in Eq. (3).

4 The three-dimensional delta-function

The notation $\delta(\mathbf{r} - \mathbf{a})$ is a short-hand form for $\delta(x - a_x)\delta(y - a_y)\delta(z - a_z)$, where x, y, z and a_x, a_y, a_z are the Cartesian coordinates of the vectors $\mathbf{r}$ and $\mathbf{a}$, respectively. Thus $\delta(\mathbf{r} - \mathbf{a})$ has dimension inverse volume and is zero for all points of space except for $\mathbf{r} = \mathbf{a}$, where it becomes infinite. In analogy to the one-dimensional case it satisfies for continuous functions $f(\mathbf{r})$

$$\int_{\mathcal{V}} \mathrm{d}^3r\, f(\mathbf{r})\delta(\mathbf{r} - \mathbf{a}) = \begin{cases} f(\mathbf{a}) & \text{if } \mathbf{a} \text{ is inside } \mathcal{V} \\ 0 & \text{if } \mathbf{a} \text{ is outside } \mathcal{V} \end{cases} \tag{4}$$

Note, that $\delta(\mathbf{r}) \neq \delta(r)$ (where $r = |\mathbf{r}|$), albeit both expressions have a spike at the origin. This can be easily seen by the fact that the left-hand side has dimension inverse volume, while the right-hand side has dimension inverse length.

Example from vector analysis
An important example is the divergence of the radial vector field $\mathbf{F}(\mathbf{r}) = F_r\mathbf{e}_r$ with $F_r = 1/r^2$, which is, e.g., proportional to the electric field of a positive point charge. We find

$$\nabla \cdot \left(\frac{\mathbf{r}}{r^3}\right) = 4\pi\delta(\mathbf{r}) \tag{5}$$

This can be shown in two ways:

(i) For $\mathbf{r} \neq 0$ we find in spherical coordinates $\nabla \cdot \mathbf{F}(\mathbf{r}) = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 F_r = 0$. On the other hand we have for a sphere $\mathcal{B}$ around the origin with arbitrary radius R

$$\int_{\mathcal{B}} \mathrm{d}^3r\, \nabla \cdot \mathbf{F}(\mathbf{r}) = \int_{\partial\mathcal{B}} \mathrm{d}\mathbf{A} \cdot \mathbf{F}(\mathbf{r}) = 4\pi R^2 F_r(R) = 4\pi$$

where we used Gauss' theorem. Thus $\frac{1}{4\pi}\nabla \cdot \mathbf{F}(\mathbf{r})$ satisfies the definition for the three-dimensional δ-function (4). This proves the conjecture (5). However, one might argue, whether Gauss' theorem is applicable to functions with a singularity, so that this motivation is not entirely clean.

(ii) The auxiliary function

$$\mathbf{F}_\epsilon(\mathbf{r}) = \frac{1}{r^3+\epsilon^3}\mathbf{r}$$

approaches $\mathbf{F}(\mathbf{r})$ for $\epsilon \to 0$. A straightforward evaluation in spherical coordinates provides

$$\nabla \cdot \mathbf{F}_\epsilon(\mathbf{r}) = 4\pi\delta_\epsilon(\mathbf{r}) \quad \text{with } \delta_\epsilon(\mathbf{r}) = \frac{3}{4\pi}\frac{\epsilon^3}{(r^3+\epsilon^3)^2}$$

Applying spherical coordinates, we find

$$\int_{\text{entire space}} \mathrm{d}^3r\, \delta_\epsilon(\mathbf{r}) = \int_0^\infty \mathrm{d}r\, 3r^2 \frac{\epsilon^3}{(r^3+\epsilon^3)^2} = \int_0^\infty \mathrm{d}t\, \frac{\epsilon^3}{(t+\epsilon^3)^2} = 1$$

In the limit $\epsilon \to 0$ the function $\delta_\epsilon(\mathbf{r})$ vanishes for all $\mathbf{r} \neq 0$, while the spatial volume integral is always unity if the origin is covered. Thus it satisfies the definition (4) for $\delta(\mathbf{r})$ in this limit. Therefore, the conjecture (5) is obtained in the limit $\epsilon \to 0$ of $\nabla \cdot \mathbf{F}_\epsilon(\mathbf{r}) = 4\pi\delta_\epsilon(\mathbf{r})$.