1 Physics slang

The delta-function $\delta(x - x_0)$ is a function, which is zero for $x \neq x_0$ and infinity at $x = x_0$. Most importantly, it satisfies $\int_a^b dx \delta(x - x_0) = 1$ if $a < x_0 < b$. A little bit more general, it can be defined by the property, that for any well-behaved (more precisely continuous) function $f(x)$ we have

$$\int_a^b dx \delta(x - x_0)f(x) = f(x_0) \quad \text{if} \quad a < x_0 < b$$  \hspace{1cm} (1)

This characterizes the delta-function very well in practice and most physicists use such a point of view. In this context it is important to realize that one can only evaluate something if there is an integral over the argument of the delta-function, and we also see, that the delta-function cancels the dimension of the integral. Thus remember

- The dimension of the delta-function is the inverse of its argument.
- The delta-function in an expression only makes sense if an integral is available for its liquidation.

**Example:** You want to describe a standard football situation. The ball is coming from the corner-point at $t_c = 5399$ s with momentum $p_0$ (air friction neglected), hits Zlatan Ibrahimović’s head (or your personal favorite forward) at $t_0 = 5400$ s, and continues into the goal. As the contact time between head and ball is very short while the force is violent, you may describe the force on the ball by

$$F(t) = U\delta(t - t_0).$$  \hspace{1cm} (2)

As force is in Newton (N) and time in seconds (s), the dimension of $U$ must be Ns, which is a momentum. Now we calculate the momentum of the ball for $t > t_0$. According to Newton we have $\dot{p} = F(t)$ and thus

$$p(t) = p_0 + \int_{t_c}^t dt' F(t') = p_0 + U \quad \text{for} \quad t > t_0$$

We see, that we obtain a meaningful result after performing an integral over the argument $t$ of the delta function.

While this was simple, we now want to evaluate the energy $E(t)$ of the ball for $t > t_0$. We know from mechanics, that the power transferred to a body moving with velocity $v(t)$ is $P(t) = v(t) \cdot F(t)$ and thus we are tempted to write

$$E(t) = E_0 + \int_{t_c}^t dt' v(t') \cdot F(t') = E_0 + v(t_0) \cdot U \quad \text{for} \quad t > t_0$$

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However this is problematic, as the velocity is not defined at \( t_0 \), where it actually changes between \( p_0/m \) and \((p_0 + U)/m\). With other words \( v(t) \) is not continuous at \( t_0 \) and thus the use of the delta-function does not make sense in this context.

This intricacy becomes more obvious if we resolve the process in time. In reality there is a finite force (here assumed to be constant for simplicity) for a finite duration \( \epsilon \sim 1 \text{ ms} \) in time:

\[
F_\epsilon(t) = \begin{cases} 
U \epsilon & \text{for } t_0 - \epsilon/2 < t < t_0 + \epsilon/2 \\
0 & \text{otherwise}
\end{cases}
\]

Using \( F_\epsilon(t) \), we can obviously evaluate \( p(t) \) and \( E(t) \) without any conceptual problems. However, it is sometimes much easier to work with \( F(t) \) from Eq. (2), in particular if the precise time-dependence of the force is not known. This is possible if all other functions within the integral have a negligible time-dependence on the time scale \( \epsilon \) (such as the position \( r(t) \) of the ball or the camera resolution, e.g.).

From this example we conclude:

The delta function \( \delta(x-x_0) \) describes a process which only occurs within a small range \( \Delta x \) around \( x_0 \). A more detailed resolution is not required, as long as all other functions in the integral are approximately constant on the scale \( \Delta x \).

2 Mathematics: Distributions

The delta-function is not a function in a mathematical sense. It can however be defined as a limiting case for functions \( \delta_\epsilon(x) \), which are well-behaving for all \( \epsilon > 0 \), and satisfy

\[
\lim_{\epsilon \to 0} \delta_\epsilon(x) = 0 \text{ for } x \neq 0 \quad \text{and} \quad \int_{-\infty}^{\infty} dx \delta_\epsilon(x) = 1
\]

Examples for such functions are

\[
\delta_\epsilon(x) = \frac{1}{2\pi} \frac{\epsilon}{x^2 + \epsilon^2/4} \quad \text{or} \quad \delta_\epsilon(x) = \frac{1}{\epsilon \sqrt{\pi}} \exp\left(-\frac{x^2}{\epsilon^2}\right)
\]

Note that in both cases the function \( \delta_\epsilon(x) \) has the dimension of \( 1/x \). Then the integral over an expression containing the delta-function can be defined as

\[
\int_a^b dx \delta(x-x_0)f(x) = \lim_{\epsilon \to 0} \int_a^b dx \delta_\epsilon(x-x_0)f(x)
\]

which provides the previous result \( f(x_0) \), provided that \( f(x) \) is well behaved in the region around \( x_0 \). The formal theory behind this is known as distribution theory. In practice this procedure is helpful for analyzing situations, where the meaning of the delta-function is not apriori clear, as in the example for the energy transfer to the football discussed above.

3 Delta function with a more complicated argument

Quite frequently one has to evaluate expressions like \( \int_a^b dx \delta(g(x))f(x) \), where \( g(x) \) is an arbitrary function. Obviously, this picks the points \( x_i \in [a, b] \), where \( g(x_i) = 0 \). The first idea is to write

\[
\int_a^b dx \delta(g(x))f(x) = \sum_{x_i} f(x_i) \quad \text{wrong!}
\]
That this expression must be wrong can be seen by considering the dimensions. As the dimension of the delta-function is its inverse argument, the left-hand side has the dimension \([xf/g]\). In contrast, the right-hand side has the dimension \([f]\). This discrepancy in dimension, can be cured by dividing by the derivative \(g'(x)\). The correct expression is

\[
\int_a^b dx \delta(g(x))f(x) = \sum_{x_i} \frac{f(x_i)}{|g'(x_i)|} \quad \text{where the } x_i \text{ satisfy } g(x_i) = 0 \text{ with } a < x_i < b \tag{3}
\]

Note the absolute value for the derivative of \(g(x)\) in the denominator! In case the derivative \(g'(x_i)\) becomes zero, the delta-function is undefined and one has to examine the underlying physical situation carefully.

Example:

\[
\int_{-\infty}^{\infty} dx \delta(\alpha(x^2 - a^2)) \sin(kx) = \frac{\sin(ka)}{2\alpha} + \frac{\sin(-ka)}{2\alpha}
\]

The proof of Eq. (3) is straightforward. At first divide the integration-interval into parts, where \(g(x)\) is monotonously increasing or decreasing. In each interval, substitute \(y = g(x)\), and use \(dx = dy/g'(x)\). Then Eq. (1) provides the desired result for monotonously increasing parts with \(g'(x) > 0\). For monotonously decreasing parts, \(g'(x) < 0\), the upper bound in \(y\) is smaller than the lower bound. However, Eq. (1) requires the upper bound to be larger. Exchanging bounds provides a minus sign, which is taken into account by the absolute value of \(g'(x)\) in Eq. (3).

4 The three-dimensional delta-function

The notation \(\delta(r - a)\) is a short-hand form for \(\delta(x - a_x)\delta(y - a_y)\delta(z - a_z)\), where \(x, y, z\) and \(a_x, a_y, a_z\) are the Cartesian coordinates of the vectors \(r\) and \(a\), respectively. Thus \(\delta(r - a)\) has dimension inverse volume and is zero for all points of space except for \(r = a\), where it becomes infinite. In analogy to the one-dimensional case it satisfies for continuous functions \(f(r)\)

\[
\int_V d^3r f(r) \delta(r - a) = \begin{cases} f(a) & \text{if } a \text{ is inside } V \\ 0 & \text{if } a \text{ is outside } V \end{cases} \tag{4}
\]

Note, that \(\delta(r) \neq \delta(r)\) (where \(r = |r|\), albeit both expressions have a spike at the origin. This can be easily seen by the fact that the left-hand side has dimension inverse volume, while the right-hand side has dimension inverse length.

Example from vector analysis

An important example is the divergence of the radial vector field \(F(r) = F_r e_r\) with \(F_r = 1/r^2\), which is, e.g., proportional to the electric field of a positive point charge. We find

\[
\nabla \cdot \left( \frac{r}{r^3} \right) = 4\pi \delta(r) \tag{5}
\]

This can be shown in two ways:

(i) For \(r \neq 0\) we find in spherical coordinates \(\nabla \cdot F(r) = \frac{1}{r^2} \frac{\partial}{\partial \theta} r^2 F_r = 0\). On the other hand we have for a sphere \(B\) around the origin with arbitrary radius \(R\)

\[
\int_B d^3r \nabla \cdot F(r) = \int_{\partial B} dA \cdot F(r) = 4\pi R^2 F_r(R) = 4\pi
\]

where we used Gauss’ theorem. Thus \(\frac{1}{r^2} \nabla \cdot F(r)\) satisfies the definition for the three-dimensional \(\delta\)-function (4). This proves the conjecture (5). However, one might argue, whether Gauss’
Theorem is applicable to functions with a singularity, so that this motivation is not entirely clean.

(ii) The auxiliary function 
\[ F_\epsilon(r) = \frac{1}{r^3 + \epsilon^3} r \]
approaches \( F(r) \) for \( \epsilon \to 0 \). A straightforward evaluation in spherical coordinates provides
\[ \nabla \cdot F_\epsilon(r) = 4\pi \delta_\epsilon(r) \]
with \( \delta_\epsilon(r) = \frac{3}{4\pi} \frac{\epsilon^3}{(r^3 + \epsilon^3)^2} \).

Applying spherical coordinates, we find
\[ \int_{\text{entire space}} d^3r \delta_\epsilon(r) = \int_0^\infty dr 3r^2 \frac{\epsilon^3}{(r^3 + \epsilon^3)^2} = \int_0^\infty dt \frac{\epsilon^3}{(t + \epsilon^3)^2} = 1 \]
In the limit \( \epsilon \to 0 \) the function \( \delta_\epsilon(r) \) vanishes for all \( r \neq 0 \), while the spatial volume integral is always unity if the origin is covered. Thus it satisfies the definition (4) for \( \delta(r) \) in this limit. Therefore, the conjecture (5) is obtained in the limit \( \epsilon \to 0 \) of \( \nabla \cdot F_\epsilon(r) = 4\pi \delta_\epsilon(r) \).