Boltzmann, Fermi, and Bose distribution

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1 Aim

The Boltzmann and Fermi (or Bose) distribution are answers to different kind of questions. As the Fermi and Bose distribution take a similar form as the Boltzmann distribution function in the limit of high energies (nondegenerate systems), the conceptual difference is easily lost, which frequently leads to confusion. Thus the underlying concept shall be highlighted here.

2 Boltzmann distribution

Consider a general system, which can be found in different states. These states are labeled by the index i and have the energy E_i. If the system is isolated, it remains in the state it has initially been prepared in. On the other hand, if energy is exchanged with a reservoir (heat bath of temperature T), we can only give a probability P_i to find the system in state i. The fundamental question is

> What is the probability to find the system in state i?
>
> Answer: In thermal equilibrium with a heat bath of temperature T the probability is given by the *Boltzmann distribution* (or canonical distribution)
>
> $$P_i = \frac{\mathrm{e}^{-\beta E_i}}{Z} \quad \text{with} \quad \beta = \frac{1}{k_B T} \quad \text{and the partition function} \quad Z = \sum_i \mathrm{e}^{-\beta E_i}$$

Here the partition function guarantees the normalization $\sum_i P_i = 1$. In order to evaluate the expectation value of the state-dependent quantity A_i, we just use $\langle A \rangle = \sum_i A_i P_i$. In particular, we find the mean energy

$$\langle E \rangle = \sum_i E_i P_i = \frac{-1}{Z} \sum_i \frac{\partial \mathrm{e}^{-\beta E_i}}{\partial \beta} = \frac{-1}{Z} \frac{\partial Z}{\partial \beta} = -\frac{\partial \log Z}{\partial \beta}$$

3 Fermi distribution

Consider a single-particle level with energy E_{level}, which may be either empty or occupied by one electron (or any other fermion), where the Pauli principle forbids double occupation.

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What is the mean occupation f_{level} of a fermionic level?

Answer: In thermal equilibrium with a heat bath of temperature T and in contact with an electron reservoir of chemical potential μ the mean occupation is given by the

$$\textit{Fermi distribution} \quad f_{\text{level}} = \frac{1}{e^{\beta(E_{\text{level}}-\mu)}+1}$$

Proof: We have to consider two states

state $i=0$ No electron in level, energy $E_0=0$

state $i=1$ One electron in level, energy $E_1 = E_{\text{level}}-\mu$ as the particle has been taken from the reservoir where it has the energy μ. (Strictly speaking μ is defined as the change of the mean energy in the reservoir by adding a particle for fixed entropy.)

Now the probability for these two states follows the Boltzmann distribution and we find the average occupation of the level:

$$f_{\text{level}} = \frac{0e^{-\beta E_0}+1e^{-\beta E_1}}{e^{-\beta E_0}+e^{-\beta E_1}} = \frac{1}{e^{\beta(E_{\text{level}}-\mu)}+1}$$

4 Bose distribution

Consider a single particle level with energy E_{level}, which may be multiple occupied by undistinguishable particles with integer spin (bosons).

What is the mean occupation n_{level} of a bosonic level?

Answer: In thermal equilibrium with a heat bath of temperature T and in contact with a particle reservoir with chemical potential μ the mean occupation is given by the

$$\textit{Bose distribution} \quad n_{\text{level}} = \frac{1}{e^{\beta(E_{\text{level}}-\mu)}-1}$$

Proof: Similar to Fermi distribution but i runs from zero to ∞ with $E_i = i(E_{\text{level}}-\mu)$.

Note that the chemical potential used here makes sense in the case of particles with integer spin (e.g. ^{4}He atoms). A second important case are oscillators with angular frequency ω which have quantum mechanical eigenstates of energy $E_n = \hbar\omega(n+1/2)$ for $n=0,1,2\ldots$. From the Boltzmann distribution we find

$$\langle n\rangle = \frac{1}{e^{\beta\hbar\omega}-1}$$

which is identical to the Bose distribution for particles in a level with energy $E_{\text{level}}=\hbar\omega$ and vanishing chemical potential. For the case of mechanical oscillations this particle is called *phonon*. Similarly, for oscillations of the electromagnetic field (light!) the corresponding particle is called *photon*. (A variety of further such quantized bosonic excitations exist, such as magnons, polaritons, etc.)