Operator treatment of harmonic oscillator

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1 General properties

The Hamilton operator of the harmonic oscillator reads

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 \tag{1}$$

Here we want to calculate the eigenvalues in an algebraic way.

We define the *lowering operator*

$$\hat{a} = \frac{1}{\sqrt{2m\hbar\omega}}(\mathrm{i}\hat{p} + m\omega\hat{x}) \tag{2}$$

Note that, in contrast to $\hat{p}$ and $\hat{x}$, $\hat{a}$ is not Hermitian and $\hat{a}^\dagger$ is called *raising operator*. They obey the essential commutator relation

$$\boxed{[\hat{a}, \hat{a}^\dagger] = \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = 1} \tag{3}$$

We find easily

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^\dagger) \qquad \hat{p} = \sqrt{\frac{m\hbar\omega}{2}}\mathrm{i}(\hat{a}^\dagger - \hat{a}) \tag{4}$$

Inserting into Eq. (1) we obtain

$$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) \tag{5}$$

Thus, the eigenstates of the harmonic oscillator correspond to the eigenstates of the *number operator* $\hat{N} = \hat{a}^\dagger\hat{a}$. Let now $|n\rangle$ be an eigenstate of $\hat{N}$ with eigenvalue n. As $\hat{N}$ is Hermitian, n is a real number (later we will find, that $n \in \mathbb{N}$). Lets consider the state $|\beta\rangle = \hat{a}|n\rangle$ We find

$$\hat{N}|\beta\rangle = \hat{a}^\dagger\hat{a}\hat{a}|n\rangle = (\hat{a}\hat{a}^\dagger - 1)\hat{a}|n\rangle = \hat{a}n|n\rangle - |\beta\rangle = (n-1)|\beta\rangle$$

Thus $|\beta\rangle$ is a further eigenstate of $\hat{N}$ with eigenvalue $n-1$. The norm of $|\beta\rangle$ is

$$\langle\beta|\beta\rangle = \langle n|\hat{a}^\dagger\hat{a}|n\rangle = n$$

As $\langle\beta|\beta\rangle$ cannot be negative, we find that the eigenvalues of the number operator satisfy $n \geq 0$. Thus we define the normalized state $|n-1\rangle$ by

$$|n-1\rangle = \frac{1}{\sqrt{n}}|\beta\rangle = \frac{1}{\sqrt{n}}\hat{a}|n\rangle \quad \text{for} \quad n \neq 0 \tag{6}$$

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By repeating this operation we can generate a chain of states $|n\rangle, |n-1\rangle, |n-2\rangle, \ldots$ which are all eigenstates of the number operator with eigenvalues n, $n-1$, $n-2$, $\ldots$, respectively. As the eigenvalues of the number operator cannot be negative, this chain must terminate with a certain state $|n-h\rangle$ where $h \in \mathbb{N}$. The condition given in Eq. (6) implies that termination is only possible if $n-h=0$ holds. Thus we conclude, that $n=h$ is a natural number.

Analogously to Eq. (6), we can show that

$$|n+1\rangle = \frac{1}{\sqrt{n+1}}\hat{a}^\dagger|n\rangle$$

is a further normalized eigenstate of $\hat{N}$ with eigenvalue $n+1$. Thus for each n, we can generate an infinite chain of eigenstates $|n\rangle, |n+1\rangle, |n+2\rangle, \ldots$, so that all natural numbers n are possible eigenvalues of $\hat{N}$.

The eigenstates $|n\rangle$ of the number operator $\hat{N} = \hat{a}^\dagger\hat{a}$ satisfy:

$$\begin{aligned} \hat{N}|n\rangle &= n|n\rangle \qquad \text{with} \quad n \in \mathbb{N} \\ |n-1\rangle &= \frac{1}{\sqrt{n}}\hat{a}|n\rangle \qquad \text{for} \quad n \neq 0 \\ |n+1\rangle &= \frac{1}{\sqrt{n+1}}\hat{a}^\dagger|n\rangle \end{aligned}$$

Simultaneously, the states $|n\rangle$ are the eigenstates of the harmonic oscillator

$$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$$

with energies $\hbar\omega(n+\frac{1}{2})$, respectively.

2 Heisenberg picture

In the Heisenberg picture the time dependence of the operators is given by

$$\frac{\mathrm{d}\hat{O}(t)}{\mathrm{d}t} = \frac{\mathrm{i}}{\hbar}[\hat{H}, \hat{O}]$$

if the operator has no explicit time-dependence. Thus we find for the lowering operator :

$$\frac{\mathrm{d}\hat{a}(t)}{\mathrm{d}t} = \mathrm{i}\omega[\hat{a}^\dagger\hat{a}, \hat{a}] = -\mathrm{i}\omega\hat{a} \qquad \Rightarrow \qquad \hat{a}(t) = \hat{a}(0)\mathrm{e}^{-\mathrm{i}\omega t}$$

and similarly $\hat{a}^\dagger(t) = \hat{a}^\dagger(0)\mathrm{e}^{\mathrm{i}\omega t}$

3 Coherent States

The energy eigenstates $|n\rangle$ provide the expectation values (Hint: use Eq. (4) and the orthonormality for different eigenstates of $\hat{H}$)

$$\langle n|\hat{x}|n\rangle = 0 \quad \text{and} \quad \langle n|\hat{p}|n\rangle = 0$$

Thus they do not resemble a classical oscillation, where x and p oscillate in time. However there are states $|\Psi(t)\rangle$ which resemble the classical picture in a much better way. A very interesting class of such states are the coherent states (or Glauber states[2])

$$|\alpha\rangle = \sum_n \mathrm{e}^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \quad \text{for arbitrary complex } \alpha \tag{7}$$

These states are actually eigenstates of the lowering operator

$$\hat{a}|\alpha\rangle = \sum_n \mathrm{e}^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}} \sqrt{n} |n-1\rangle = \alpha \sum_n \mathrm{e}^{-|\alpha|^2/2} \frac{\alpha^{n-1}}{\sqrt{(n-1)!}} |n-1\rangle = \alpha|\alpha\rangle \tag{8}$$

with eigenvalue α. (Note that $\hat{a}$ is not Hermitian. Thus, eigenstates to different eigenvalues α are neither orthogonal, nor do they satisfy the closure relation.) As the coherent states are not eigenstates of the Hamiltonian, they have a more complicated time dependence. For the initial condition $|\Psi(t=0)\rangle = |\alpha_0\rangle$ we find

$$\begin{aligned} |\Psi(t)\rangle =& \sum_n \mathrm{e}^{-|\alpha_0|^2/2} \frac{\alpha_0^n}{\sqrt{n!}} \mathrm{e}^{-\mathrm{i}\left(n+\frac{1}{2}\right)\omega t} |n\rangle = \mathrm{e}^{-\mathrm{i}\omega t/2} \sum_n \mathrm{e}^{-\left|\alpha_0 \mathrm{e}^{-\mathrm{i}\omega t}\right|^2/2} \frac{\left(\alpha_0 \mathrm{e}^{-\mathrm{i}\omega t}\right)^n}{\sqrt{n!}} |n\rangle \\ =& \mathrm{e}^{-\mathrm{i}\omega t/2} |\alpha(t)\rangle \quad \text{with } \alpha(t) = \alpha_0 \mathrm{e}^{-\mathrm{i}\omega t} \end{aligned} \tag{9}$$

Thus an initial coherent state is also a coherent state for later times with a change in the phase of α (as well as a multiplicative phase $\mathrm{e}^{-\mathrm{i}\omega t/2}$, which is not of relevance for any observable).

In order to calculate the expectation values of space and momentum, we use Eq. (4) and $\langle\alpha|\hat{a}^\dagger = \alpha^*\langle\alpha|$. Then we find

$$\langle\alpha|\hat{x}|\alpha\rangle = \sqrt{\frac{\hbar}{2m\omega}}(\alpha^* + \alpha) = \sqrt{\frac{2\hbar}{m\omega}} \operatorname{Re}\{\alpha\} \tag{10}$$

$$\langle\alpha|\hat{p}|\alpha\rangle = \mathrm{i}\sqrt{\frac{\hbar m\omega}{2}}(\alpha^* - \alpha) = \sqrt{2\hbar m\omega} \operatorname{Im}\{\alpha\} \tag{11}$$

With the time dependence $\alpha(t) = \alpha_0 \mathrm{e}^{-\mathrm{i}\omega t}$, the averages reproduce the classical trajectories in phase space, as can be seen in Fig. 1.

In the same spirit (and using $\hat{a}\hat{a}^\dagger = \hat{a}^\dagger\hat{a} + 1$) we obtain

$$\langle\alpha|\hat{x}^2|\alpha\rangle = \frac{2\hbar}{m\omega}(\operatorname{Re}\{\alpha\})^2 + \frac{\hbar}{2m\omega} \tag{12}$$

$$\langle\alpha|\hat{p}^2|\alpha\rangle = 2\hbar m\omega(\operatorname{Im}\{\alpha\})^2 + \frac{\hbar m\omega}{2} \tag{13}$$

This provides the variance

$$\Delta x = \sqrt{\langle\alpha|\hat{x}^2|\alpha\rangle - \langle\alpha|\hat{x}|\alpha\rangle^2} = \sqrt{\frac{\hbar}{2m\omega}} \tag{14}$$

$$\Delta p = \sqrt{\langle\alpha|\hat{p}^2|\alpha\rangle - \langle\alpha|\hat{p}|\alpha\rangle^2} = \sqrt{\frac{\hbar m\omega}{2}} \tag{15}$$

which describe the scattering of measurement results around the expectation value as indicated in Fig. 1. We find that Δx is just the maximal elongation divided by $2|\alpha|$. Thus the relative fluctuations in the measurement results for the position vanish in the limit of larger $|\alpha|$. The

[2] R.J. Glauber, Phys. Rev. **131**, 2766-2788 (1963)

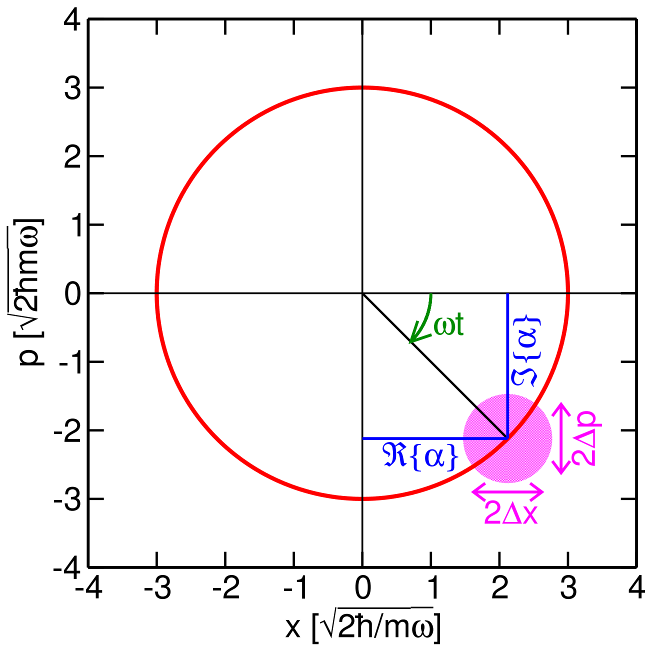

Figure 1: Motion of a Glauber state with $\alpha_0 = 3$. The red circle is the classical trajectory, which is identical to the expectation values for space and momentum. The magenta area denotes the range of typical measurement values for x and p at a given time. (Note that only one of them can be measured for each preparation of the system.)

same holds for Δp, so that we recover the classical behavior with well defined position and momentum for $|\alpha| \to \infty$. Furthermore we find the product

$$\Delta x \Delta p = \frac{\hbar}{2} \tag{16}$$

which is the lowest possible value according to the Heisenberg uncertainty relation.