6 Electrodynamics and Relativity

6.1 Basic concepts

Albert Einstein derived this theory 1905 based on the postulates

- The laws of nature are independent of the translational motion of the system as a whole (postulate of relativity).
- The speed of light \( c \) in vacuum is finite and independent of the motion of the source.

This provides specific rules if one transforms the coordinates \((x, y, z, t)\) of an event as observed in a reference frame \(\Sigma\) to the coordinates \((x', y', z', t')\) observed in a reference frame \(\Sigma'\) which moves with constant velocity \(ve_x\) with respect to \(\Sigma\) (Synchronization of clocks depending on frame)

The Lorentz transformation

\[
x' = \gamma_v(x - vt), \quad y' = y, \quad z' = z, \quad t' = \gamma_v(t - vx/c^2)
\]

with \(\gamma_v = \frac{1}{\sqrt{1 - v^2/c^2}} > 1\)

Discuss, show that \(c\) is invariant in \(x\) and \(y\) direction, compare with Galilean transformation

An important aspect is the proper time of a body moving with velocity \(u\) in its own frame of reference. We find \(\tau = t/\gamma_u\) which is less than the time seen in the frame which observes the moving body. In the same way lengths \(L_x\) are largest in the frame where they rest.

A clock in a moving system appears to go slower (time dilatation)
A ruler (in moving direction) in a moving system appears shorter (Lorentz contraction)

For example, muons \((m_0c^2 = 105.6 \text{ MeV})\) have a lifetime of approx 2.2\(\mu\)s. This is the time in their own frame. An observer seeing them moving with \(u = 99.8\%\) of the speed of light \((\gamma_u = 16)\) can observe them for \(t = \gamma_u\tau = 35\mu\)s and thus traveling an average distance of 10.5 km.\(^2\) (Alternatively one can argue that the muons see a shorter path through the atmosphere due to the Lorentz contraction.)

Assume a body has velocity \(u'\) is \(x\)-direction within the frame \(\Sigma'\). Thus the coordinates are \(x'_1\) at \(t' = t'_1\) and \(x'_2 = x'_1 + u'(t'_2 - t'_1)\) at \(t' = t'_2\). Transforming both points into the frame \(\Sigma\) yields \(x_1 = \gamma_v(x'_1 + vt'_1)\), \(x_2 = \gamma_v(x'_2 + vt'_2)\), \(t_1 = \gamma_v(t'_1 + vx'_1/c^2)\), \(t_2 = \gamma_v(t'_2 + vx'_2/c^2)\). Thus the velocity \(u\) observed in \(\Sigma\) is

\[
u = \frac{x_2 - x_1}{t_2 - t_1} = \frac{u' + v}{1 + \frac{vu'}{c^2}}
\]

If \(u'\) is not parallel to \(v\), the result is slightly more complicated, see Eq. (11.31) of Jackson.

Compare with Galilean transformation, limited by \(c\)

If we had a signal being transformed form source to observer with a velocity \(u > c\), we could switch to a new reference frame with the negative velocity satisfying \(c^2/u < -v < c\). In this frame the signal would have a negative velocity violating causality. Thus information cannot travel with superluminal velocities.

\(^2\)Thus such muons produced by cosmic radiation in the upper atmosphere can be detected on earth. B. Rossi and D. B. Hall 1941 Phys. Rev. 59, 223 (1941)
6.2 Mathematical formulation of Space-Time

For convenience we define in a given frame the contravariant components \( x^\nu \) by \( x^0 = ct, x^1 = x, x^2 = y, x^3 = z \). These can be viewed as a vector \( \hat{x} = x^\nu \hat{e}_\nu \) in the 4-dimensional vector space of space-time (these vectors are written as \( \hat{a} \) in order to distinguish them from \( a \) in 3 dimensional Euclidean space) where the Einstein sum convention (sum over indices \( \nu = 0, 1, 2, 3 \) if they appear in pairs of an upper and lower index) is applied. We define the bilinear form \( \hat{e}_\nu \cdot \hat{e}_\mu = g_{\nu\mu} \) with the metric tensor

\[
g_{\nu\mu} = g^{\nu\mu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}
\]

Thus we obtain \( \hat{x} \cdot \hat{y} = x^\nu g_{\nu\mu} y^\mu = x^0 y^0 - x^1 y^1 - x^2 y^2 - x^3 y^3 \) The Lorentz transformations defined above are mappings \( \hat{x} \rightarrow \hat{x}' \) which are represented by the matrix

\[
\Lambda^\nu_\mu = \begin{pmatrix} \gamma_v & -\gamma_v \beta & 0 & 0 \\ -\gamma_v \beta & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]

so that \( x'^\nu = \Lambda^\nu_\mu x^\mu \). These transformations satisfy \( \hat{x}' \cdot \hat{y}' = \hat{x} \cdot \hat{y} \) and so do rotations and reflections in space, which all together with their products form form the Lorentz group. Adding translations in space and time, one obtains the Poincaré group, the symmetry group of free space.

A set of variables \( a^\nu \) which satisfies a transformation property \( a'^\nu = \Lambda^\nu_\mu a^\mu \) for the Lorentz-transformations is called a tensor. They can be contravariant as in the example above or covariant, where \( a'_\mu = (\Lambda^{-1})^\nu_\mu a_\nu \). In the case of Lorentz transformations one can verify \( (\Lambda^{-1})^\nu_\mu g^\rho_\nu \Lambda^\rho_\kappa g_{\kappa\mu} \). Thus the objects \( a_\nu = g_{\nu\rho} a^\rho \) are covariant components. Furthermore the definition of tensors of second order is similar, which can be purely contravariant \( A^\nu_\mu \), purely covariant \( A_{\nu\mu} \), or mixed \( A^\nu_\mu / A_\nu^\mu \). We find that the derivative \( \frac{\partial}{\partial x^\nu} = \partial_\nu \) is a covariant tensor.

In order to show their invariance, we have to express physical quantities in such a way that their transformation behavior for Lorentz transformations becomes clear. While the standard three dimensional velocity \( u = dr/dt \) of a body transforms in a difficult way, the four vector \( U^\nu = dx^\nu /d\tau \), where \( \tau \) is the proper time of the moving body, is a contravariant tensor. The corresponding relativistic momentum reads \( \hat{p} = m \hat{U} = (m\gamma_u c, m\gamma_u u_0) \).

For elastic scattering processes it is shown in Section 11.5 of Jackson that \( \hat{p} = m\gamma_u \hat{u} \) and \( E_{\text{kin}} = m\gamma_u c^2 - mc^2 \) are the momentum and kinetic energy, which are conserved in the collision. For inelastic processes, consider the decay of a neutral pion into an electron and a positron. In the frame of the pion \( \Sigma \) the velocities of the electron and positron are \( \pm u \), and their kinetic energy is \( E_{\text{kin}} = m_e\gamma_e c^2 - m_e c^2 \). The mass of the pion can then be determined via the principle of relativity in the frame \( \Sigma' \) moving with velocity \( v_\Sigma e_\Sigma \) with respect to \( \Sigma \). After the decay we have

\[
u'_e = \frac{u - v}{1 - \frac{uv}{c^2}}, \quad u'_p = -\frac{u + v}{1 + \frac{uv}{c^2}} \quad \rightarrow \quad p^\Sigma_{\text{sum}} = m_e\gamma_e u'_e + m_e\gamma_e u'_p = 2m_e\gamma_e u'_e v
\]

Momentum conservation in \( \Sigma' \) means that the pion must have had the same momentum before the collision. As the velocity \( v \) is known, its mass must be \( m_{\text{Pion}} = 2m_e\gamma_e = 2E_{\text{kin}}/c^2 + 2m_e \). The mass difference \( m_{\text{Pion}} - 2m_e \) is transformed into kinetic energy and \( E = m\gamma_u c^2 = p^0 c \) can be interpreted as the total energy of a moving particle. This provides the relativistic relation between energy and momentum \( p_u p^u = E^2/c^2 - p^2 = m^2 c^2 \).
6.3 Covariant formulation of Electrodynamics

The current density for a point charge is related to the velocity and should thus transform as a contravariant tensor \( j^\nu \), if one sets \( j^0 = cp \). Then the equation of continuity becomes \( \partial_\nu j^\nu = 0 \), which is indeed invariant in all frames. A collection of important tensors is:

<table>
<thead>
<tr>
<th>4-vector</th>
<th>Important equations</th>
</tr>
</thead>
<tbody>
<tr>
<td>velocity ( U^\nu = dx^\nu/d\tau ) ( \mathbf{U} = (\gamma_0 c, \gamma_0 \mathbf{u}) ) ( p^\nu = mU^\nu ) ( \mathbf{p} = (E/c, \mathbf{p}) ) ( \nu = 0 ) ( j^\nu \mathbf{j} ) ( \nu )∂, ( \nu \mu )∂ ( \partial_\nu A^\nu = \mu_0 j^\nu ) for Lorenz gauge ( \partial_\nu A^\nu = 0 )</td>
<td></td>
</tr>
<tr>
<td>potential ( \mathbf{A} = (\phi/c, \mathbf{A}) ) ( \nu )∂, ( \nu \mu )∂ ( \partial_\nu A^\nu = \mu_0 j^\nu ) for Lorenz gauge ( \partial_\nu A^\nu = 0 )</td>
<td></td>
</tr>
<tr>
<td>Field strength tensor ( F^\nu{}^\mu = \partial^\nu A^\mu - \partial^\mu A^\nu ) ( \begin{pmatrix} 0 &amp; -E_x/c &amp; -E_y/c &amp; -E_z/c \ E_x/c &amp; 0 &amp; -B_z &amp; B_y \ E_y/c &amp; B_z &amp; 0 &amp; -B_x \ E_z/c &amp; -B_y &amp; B_x &amp; 0 \end{pmatrix} ) ( \nu \mu )∂</td>
<td>Lorentz force ( \nu )∂, ( \nu \mu )∂ ( \nu )∂, ( \nu \mu )∂ ( \partial_\nu A^\nu = \mu_0 j^\nu ) for Lorenz gauge ( \partial_\nu A^\nu = 0 )</td>
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6.4 Transformation of plane electromagnetic waves

Consider a wave train of the form \( \mathbf{E}(r, t) = E_0 e_\mathbf{e}_x f(x - ct) \), \( \mathbf{B}(r, t) = \frac{E_0}{c} e_z f(x - ct) \). In the frame \( \Sigma' \) moving with velocity \( \nu \) in \( e_x \) direction, we have to replace \( x = \gamma_\nu (x' - \nu t') \), \( t = \gamma_\nu (t' - \nu x'/c^2) \) and obtain the field strength tensor

\[
F'_{\nu\mu} = \gamma_\nu \gamma_\mu \gamma_\nu', \gamma_\mu' \gamma_{\nu'} = \begin{pmatrix} 0 & 0 & 0 & -\frac{E_0}{c} f'(x' - ct') \\ 0 & 0 & -\frac{E_0}{c} f'(x' - ct') & 0 \\ 0 & -\frac{E_0}{c} f'(x' - ct') & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ with } f'(\xi) = f[\gamma_\nu (1 - \beta)\xi] \]

\[
E'_0 = \gamma_\nu (1 - \beta) E_0 \]

Thus a plane wave \( f(\xi) = e^{i\omega\xi/c} \) with frequency \( \omega \) in \( \Sigma \) is observed as a plane wave with frequency \( \omega' = \gamma_\nu (1 - \beta)\omega \) in \( \Sigma' \). The energy in the electromagnetic field is given by \( U = \oint d^3r \left( \frac{\omega}{2} \mathbf{E}^2 + \frac{1}{2\mu_0} \mathbf{B}^2 \right) \). For a finite area \( A \) of the wave in \( y, z \) direction we find \( U = \frac{\omega}{2} |E_0|^2 AL \) and \( U' = \frac{\omega'}{2} |E'_0|^2 AL' = \gamma_\nu (1 - \beta) U \). Thus the energy of the pulse transforms like the frequency as already remarked by Einstein in his very first article on the subject. Discuss quantization.