Lunds Universitet
Tineke van den Berg, Johannes Berlin

Spring 2016

Exercises to Quantum Mechanics FYSN17/FMFN01, Week 1

Homework to be handed in until January 26 in Johannes' mailbox

Exercise 1 (Homework): *Commutator Relations*

Aim: Learning to work with commutators

a) Prove the commutator relation: $[\hat{A}, \hat{B}\hat{C}] = [\hat{A}, \hat{B}]\hat{C} + \hat{B}[\hat{A}, \hat{C}]$.

b) Calculate the commutator: $[\hat{p}_x^2, \hat{x}^2]$.

c) Show by induction that $[\hat{B}, \hat{A}^n] = n[\hat{B}, \hat{A}]\hat{A}^{n-1}$ if $\big[[\hat{B}, \hat{A}], \hat{A}\big] = 0$.

Exercise 2 (Homework): *Common eigenstates*

Aim: Familiarity with an important rule, which actually holds in both ways, and is frequently applied

Two operators have a set of common eigenstates, which is complete in the ket space. Prove that the operators commute.

Exercise 3 (Homework): *Measurement*

Aim: Applying the probability interpretation of quantum mechanics

Let $|\psi_1\rangle$ and $|\psi_2\rangle$ be two orthogonal normalized states of a physical system:

$$\text{i.e.} \quad \langle\psi_1|\psi_1\rangle = \langle\psi_2|\psi_2\rangle = 1 \quad \text{and} \quad \langle\psi_1|\psi_2\rangle = 0$$

and let $\hat{A}$ be an observable of the system. Consider a nondegenerate eigenvalue of $\hat{A}$ denoted by α_n to which the normalized state $|\phi_n\rangle$ corresponds. We define

$$P_1(\alpha_n) = |\langle\phi_n|\psi_1\rangle|^2$$
$$P_2(\alpha_n) = |\langle\phi_n|\psi_2\rangle|^2$$

a) What is the interpretation of $P_1(\alpha_n)$ and $P_2(\alpha_n)$?

b) A given particle is in the state $|\psi\rangle = 3|\psi_1\rangle - 4\mathrm{i}|\psi_2\rangle$. What is the probability of getting α_n when $\hat{A}$ is measured?

Exercise 4: *Eigenstates in matrix representation*

Aim: Familiarity with basic operator properties and matrix representation

In a three dimensional subspace two operators are represented by the matrices:

$$\hat{A} = \begin{bmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{bmatrix}, \hat{B} = \begin{bmatrix} b & 0 & 0 \\ 0 & 0 & ib \\ 0 & -ib & 0 \end{bmatrix}$$

a and b are real numbers.

a) Show that $\hat{A}$ and $\hat{B}$ are Hermitian.

b) Calculate the eigenvalues of $\hat{A}$ and $\hat{B}$.

c) Show that $[\hat{A}, \hat{B}] = 0$.

d) Determine a basis of common eigenstates.

See other side

Exercise 5: *Properties of the adjoint operator*

Aim: Familiarity with important relations for the adjoint operator

Derive the following properties of the adjoint of an operator:

a) $(\hat{A}^\dagger)^\dagger = \hat{A}$;

b) $(\lambda\hat{A})^\dagger = \lambda^*\hat{A}^\dagger$, where λ is a complex number;

c) $(\hat{A}+\hat{B})^\dagger = \hat{A}^\dagger + \hat{B}^\dagger$;

d) $(\hat{A}\hat{B})^\dagger = \hat{B}^\dagger\hat{A}^\dagger$.

Exercise 6: *Operator properties (facultative)*

Aim: A challenge for more mathematical interested students

Consider a Hermitian operator $\hat{A}$ that has the property that $\hat{A}^3 = \mathbf{1}$. Show that $\hat{A} = \mathbf{1}$. (**Hint:** Look at the eigenstates of $\hat{A}$!)

Exercise 7: *Closure relation*

Aim: Familiarity with an important relation, we will often use

Prove that if an orthonormal discrete set of kets $\{|u_i\rangle,\ i = 1, 2, \ldots\}$ constitutes a basis, then it follows that

$$\sum_i |u_i\rangle\langle u_i| = \mathbf{1}$$

Exercise 8: *Time-evolution of stationary states*

Aim: Apply the equation of motion to a particle in a well

A particle of mass m is confined within an infinite one-dimensional well, between $x = 0$ and $x = a$. The stationary states $|\phi_n\rangle$ of the particle have energies

$$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}, \quad n = 1, 2, 3, , ...$$

and to the wavefunction $\phi_n(x) = \sqrt{\frac{2}{a}}\sin(\frac{n\pi}{a}x)$. Consider the case in which at time t=0 the particle is in the state $|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|\phi_n\rangle + |\phi_2\rangle)$.

a) Find the time-dependent $|\psi(t)\rangle$.

b) Calculate the wave function $\psi(x,t)$.

The following problems belong to chapter 1 but are done in week 2

Exercise 9: *Time evolution*

Aim: Get into contact with an important explicit solution for the Schrödinger equation

Show that the unitary operator

$$\hat{U}(t_1, t_0) = \exp\left(\frac{\hat{H}}{\mathrm{i}\hbar}(t_1 - t_0)\right)$$

maps $|\Psi(t_0)\rangle$ onto $|\Psi(t_1)\rangle$ if $|\Psi(t)\rangle$ satisfies the Schrödinger equation with a time-independent Hamilton operator $\hat{H}$.

Exercise 10: *Time-evolution for the hydrogen atom*

Aim: Learn to apply the equation of motion for expectation values

Consider an electron moving in the potential $V(\mathbf{r}) = -\frac{e^2}{4\pi\epsilon_0|\mathbf{r}|}$ of a resting proton. Provide the Hamilton operator and determine

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\Psi|\hat{p}_x|\Psi\rangle$$

Compare the result with the classical case!