Lunds Universitet
Tineke van den Berg, Johannes Berlin

Spring 2016

Exercises to Quantum Mechanics FYSN17/FMFN01, Week 2

Homework to be handed in until February, 2 in Johannes' mailbox

The first seven problems deal with the one-dimensional harmonic oscillator with Hamiltonian $\hat{H}_{\text{oscil}} = \hat{p}^2/2m + m\omega^2\hat{x}^2/2$ and eigenstates $|\phi_n\rangle$ with energy $(n+1/2)\hbar\omega$, where $n = 0, 1, 2, \ldots$. The operators $\hat{a}$ and $\hat{a}^\dagger$ are the corresponding step operators.

Exercise 1: ***Matrix elements***

Aim: Training the use of step operators

Calculate the matrix elements $\langle\phi_n|\hat{p}^2|\phi_l\rangle$

Exercise 2: ***Kinetic and potential energy***

Aim: Training the use of step operators; seeing an example for the virial theorem

Calculate the expectation values of the kinetic and potential energy

$$\langle\hat{T}\rangle = \left\langle\frac{\hat{p}^2}{2m}\right\rangle \quad \text{and} \quad \langle\hat{V}\rangle = \left\langle\frac{m\omega^2\hat{x}^2}{2}\right\rangle$$

if the oscillator is in its n^{th} eigenstate $|\phi_n\rangle$.

Hint: Write $\hat{p}$ and $\hat{x}$ in terms of $\hat{a}^\dagger$ and $\hat{a}$.

Exercise 3 (Homework): ***Uncertainty Relation***

Aim: Training the use of step operators; reminding on uncertainty from previous courses

Calculate the uncertainty product $\Delta x \cdot \Delta p$ for the eigenstate $|\phi_n\rangle$.

Hint: Remember that

$$\Delta x = \sqrt{\langle\hat{x}^2\rangle - \langle\hat{x}\rangle^2}, \quad \Delta p = \sqrt{\langle\hat{p}^2\rangle - \langle\hat{p}\rangle^2}$$

Exercise 4 (Homework): ***Time-dependence of expectation values***

Aim: Applying general formula; getting familiar with classical correspondence

Solving the stationary Schrödinger equation in a finite basis by diagonalization Assume the oscillator is in a state $|\Psi(t)\rangle$ satisfying the Schrödinger equation.

a) Calculate $\langle\Psi(t)|\hat{a}^\dagger|\Psi(t)\rangle$ and $\langle\Psi(t)|\hat{a}|\Psi(t)\rangle$ as a function of time for given initial values $a_0 = \langle\Psi(0)|\hat{a}^\dagger|\Psi(0)\rangle$ and $b_0 = \langle\Psi(0)|\hat{a}|\Psi(0)\rangle$.

b) Determine $\langle\Psi(t)|\hat{x}|\Psi(t)\rangle$ and $\langle\Psi(t)|\hat{p}|\Psi(t)\rangle$ from the result of a).

Hint: Use$\frac{d\langle\Psi|\hat{A}|\Psi\rangle}{dt} = \frac{i}{\hbar}\langle\Psi|[\hat{H}, \hat{A}]|\Psi\rangle$.

Exercise 5: ***Oscillator in an electric field***

Aim: Training to solve the stationary Schrödinger equation by diagonalization

Consider the Hamiltonian $\hat{H} = \hat{H}_{\text{oscil}} - qF\hat{x}$, which describes the harmonic oscillator with charge q with an additional electric field F. Consider the subspace spanned by $|\phi_0\rangle$ and $|\phi_1\rangle$ and calculate the eigenstates and eigenenergies of $\hat{H}$ in this subspace.

Hint: Write $\hat{x}$ in terms of $\hat{a}^\dagger$ and $\hat{a}$.

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Exercise 6: *Coherent state (facultative)*

Aim: Getting familiar with states which are central for quantum optics

A so called coherent state is described by a wavefunction of the form:

$$|\phi_\beta\rangle = C \sum_{n=0}^{\infty} \frac{\beta^n}{n!} (\hat{a}^\dagger)^n |\phi_0\rangle$$

a) Show that $|\phi_\beta\rangle$ is an eigenstate of the operator $\hat{a}$ and determine the eigenvalue.

b) Determine $C = C(t)$ and $\beta = \beta(t)$ so that $|\phi_\beta\rangle$ satisfies the time dependent Schrödinger equation.

Hints: a) Write $\hat{a}(\hat{a}^\dagger)^n$ as a normal ordered product (all $\hat{a}^\dagger$ terms precede $\hat{a}$) b) Plug $|\phi_\beta\rangle$ into the time-dependent Schrödinger equation and compare coefficients.

The following problems deal with harmonic oscillators of higher dimensions.

Exercise 7: *Angular momentum*

Aim: Training the application of step operators in three dimensions

Consider a three-dimensional harmonic oscillator.

a) Show that the operator $\hat{L}_x = (\hat{r} \times \hat{p})_x$ can be expressed as $\hat{L}_x = i\hbar(\hat{a}_z^\dagger \hat{a}_y - \hat{a}_y^\dagger \hat{a}_z)$.

b) Calculate the commutator $[\hat{H}, \hat{L}_x]$.

Exercise 8: *Magic numbers of isotropic oscillator*

Aim: Learning the concept of shell structure and magic numbers

Assume that we have non-interacting fermions with spin 1/2 in a three-dimensional harmonic oscillator. Such particles obey the Pauli exclusion principle and you can put at most two particles (one with spin up and one with spin down) into each single-particle eigenstate (called level). Now you fill N particles into the levels such that the total energy is minimal, i.e. the levels with the lowest possible energies are occupied. Determine the *magic numbers* in such a system.

Explanation: N is called a *magic number*, if the $(N+1)$th fermion would need to occupy a level with an energy, which is larger than the energy of any level occupied for N fermions. (In this case the configuration with N particles is particularly stable, such as the noble gases in the periodic table.)

Exercise 10 (Homework): *Magic numbers of anisotropic oscillator*

Aim: Applying the concept of shell structure and magic numbers

In the lecture notes we only considered isotropic oscillators. An anisotropic two-dimensional oscillator is given by the Hamiltonian

$$\hat{H} = \frac{(\hat{p}_x^2 + \hat{p}_y^2)}{2m} + \frac{m}{2}(\omega_x^2 \hat{x}^2 + \omega_y^2 \hat{y}^2)$$

where $\omega_x \neq \omega_y$. Consider the special case $\omega_y = 2\omega_x$. Calculate the eight lowest energy levels and determine the magic numbers for the case of fermions.