Lunds Universitet
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Spring 2016

Exercises to Quantum Mechanics FYSN17/FMFN01, Week 3

Homework to be handed in until February, 9 in Johannes' mailbox

Exercise 1: *Rotational symmetry*

Aim: Getting familiar with commutation relations for $\hat{L}$

A potential $V(\mathbf{r})$ is rotational symmetric with respect to two perpendicular axes. Show that it must be spherical symmetric.

Hint: Show that $[V(\hat{\mathbf{r}}), \hat{L}_x] = 0$ and $[V(\hat{\mathbf{r}}), \hat{L}_y] = 0 \Rightarrow [V(\hat{\mathbf{r}}), \hat{L}_z] = 0$.

Exercise 2: *Properties of operators with rotational symmetry*

Aim: Understanding central relations for operators with rotational symmetry

$\hat{A}$ is an operator that commutes with all components of $\hat{\mathbf{J}}$. According to Chapter 3.1. of the lecture this reflects its rotational symmetry. Show that:

a) $\langle j^{'} m^{'} | \hat{A} | jm \rangle = 0$ when $m \neq m^{'}$ or $j \neq j^{'}$.

b) $\langle jm | \hat{A} | jm \rangle$ is independent of m. Hint: use the step operators!

Exercise 3 (Homework): *Operators*

Aim: Training the use of commutation relations

Suppose that there are three operators $(\hat{T}_x, \hat{T}_y, \hat{T}_z)$ that satisfy the following commutation relations with the angular momentum operators $(\hat{J}_x, \hat{J}_y, \hat{J}_z)$

$$[\hat{J}_x, \hat{T}_x] = 0\,, \quad [\hat{J}_x, \hat{T}_y] = i\hbar\hat{T}_z \quad \text{and} \quad [\hat{J}_x, \hat{T}_z] = -i\hbar\hat{T}_y$$

and cyclic permutations. Furthermore we define $\hat{T}_+ = \hat{T}_x + i\hat{T}_y$

a) Prove the following commutation relations

$$[\hat{J}_z, \hat{T}_+] = \hbar\hat{T}_+ \quad \text{and} \quad [\hat{\mathbf{J}}^2, \hat{T}_+] = 2\hbar^2\hat{T}_+ + 2\hbar(\hat{T}_+\hat{J}_z - \hat{T}_z\hat{J}_+)$$

b) Show that $\hat{T}_+|j, j\rangle = c|j+1, j+1\rangle$ where $|j, m\rangle$ are the common eigenstates of the angular momentum operators $\hat{J}_z$, $\hat{\mathbf{J}}^2$ and c is a constant.

Exercise 4: *Matrix representation for* $J = 3/2$

Aim: Understanding the fundamental structure of multiplets

Determine the matrices for $\hat{J}_x, \hat{J}_y$ and $\hat{J}_z$ in $\mathcal{M}^{\frac{3}{2}}$. Check that the matrices satisfy the commutation relation: $[\hat{J}_x, \hat{J}_y] = i\hbar\hat{J}_z$.

Exercise 5: *Angular momentum measurement for a wave function*

Aim: Applying the measurement concept in quantum mechanics; working with spherical harmonics

The wave function for a particle is given by

$$\Psi(\mathbf{r}) = f(r)(\sin\theta\cos\varphi + \cos\theta + 1)$$

in spherical coordinates, where $f(r)$ is an arbitrary function satisfying the normalization $\int \mathrm{d}^3r |\Psi(\mathbf{r})|^2 = 1$. One measures $\hat{L}^2$ and $\hat{L}_z$ simultaneously. What are the possible outcomes of the measurement and what are the probabilities of each outcome?

Exercise 6 (Homework): *Energy of an ion with spin*

Aim: Working with matrix representations; calculation of energy eigenstates

An ion in a crystal lattice has an effective spin $= \hbar$. Suppose that the only degree of freedom is the spin. The spin interacts with the surrounding lattice so that: $\hat{H}_{spin} = A\hat{S}_z^2$.

a) Write $\hat{H}_{spin}$ as a matrix.

b) Determine the energy eigenvalues.

c) An external magnetic field $\mathbf{B} = B\mathbf{e}_z$ is switched on. Calculate the new energies if the ion has a magnetic moment $\hat{\mu} = \gamma\hat{\mathbf{S}}$, where $\hat{\mathbf{S}} = \hat{S}_x\mathbf{e}_x + \hat{S}_y\mathbf{e}_y + \hat{S}_z\mathbf{e}_z$

d) The crystal is turned so that $\mathbf{B} = B\mathbf{e}_x$. What are the new energies?

Exercise 7: *Products of Pauli matrices*

Aim: Getting familiar with Pauli matrices

Show by direct computation the following formulae:

$$\begin{aligned} \sigma_x^2 = \sigma_y^2 = \sigma_z^2 =& \mathbf{1} & \qquad \sigma_x\sigma_y =& i\sigma_z \\ \sigma_y\sigma_z + \sigma_z\sigma_y =& 0 & \qquad \sigma_x\sigma_y\sigma_z =& i\mathbf{1} \end{aligned}$$

Exercise 8: *Finite rotation of a spin 1/2 system*

Aim: Working with operator and representations; Concept of unitarity; Important feature of spin

Let $\hat{S}_x$ be the spin operator for a spin$\frac{1}{2}$-particle and $\hat{U}$ the operator

$$\hat{U} = \exp\left(-\frac{i}{\hbar}\theta\hat{S}_x\right).$$

a) Show that the representation of $\hat{U}$ in $\mathcal{M}^{\frac{1}{2}}$ is given by:

$$\hat{U} = \cos\left(\frac{\theta}{2}\right)\mathbf{1} - i\sin\left(\frac{\theta}{2}\right)\sigma_x.$$

b) Prove that $\hat{U}$ is unitary.

c) The operator $\hat{U}$ is providing a rotation around the x axis by the angle θ. Discuss the action of this rotation for angles of $\theta = 2\pi$ and $\theta = 4\pi$.