Lunds Universitet
Tineke van den Berg, Johannes Bjerlin Spring 2016

Exercises to Quantum Mechanics FYSN17/FMFN01, Week 6

Homework to be handed in on March 1

Exercise 1: *Level crossing*

Aim: Applying stationary perturbation theory; understanding a generic scenario

There is a rule in quantum mechanics that says: 'Perturbations remove level-crossings'. A level-crossing is when the Hamiltonian depends on a parameter and the eigenvalues of it cross if plotted as functions of this parameter. We shall investigate this in a very simple model. Consider a two dimensional state space where the unperturbed Hamiltonian $\hat{H}_0$ and the perturbation $\hat{H}_p$ are given by:

$$\hat{H}_0(\lambda) = \begin{bmatrix} k\lambda & 0 \\ 0 & -k\lambda \end{bmatrix}, \hat{H}_p = \begin{bmatrix} 0 & c \\ c^* & 0 \end{bmatrix} \tag{1}$$

Here k is a positive real constant, c a constant, and λ a parameter $(-1 < \lambda < 1)$.

a) Plot the eigenvalues of $\hat{H}_0$ as a function of λ.

b) Calculate the eigenvalues of $\hat{H}_0 + \hat{H}_p$ and plot those in the same diagram.

c) Treat the system in perturbation theory and show that the second order-approximation becomes bad, when λ approaches 0.

Exercise 3: *Optical transition*

Aim: Learning how to apply the time-dependent perturbation theory

Consider a one-dimensional harmonic oscillator with frequency ω_0 and mass m. For $t \to -\infty$ the oscillator is in its ground state. Calculate the transition probability to the first excited state for the perturbation $\hat{V}(t) = F\hat{x}\mathrm{e}^{-t^2/\tau^2}$.

Hint: replace the integration limits in Eq. (5.20) of the compendium by $\pm\infty$.

Exercise 4 (Homework): *Rabi Oscillation in a quantum dot*

Aim: Training time-dependent perturbation theory; comparing with exact result

A quantum dot shall have two energy levels $|1\rangle$ with $E_1^0 = 0$ and $|2\rangle$ with $E_2^0 = \Delta E > 0$. A laser field provides a periodic perturbation $\hat{V}(t) = 2\hat{F}\cos(\omega t)$, which is in resonance with the level spacing, i.e., $\Delta E = \hbar\omega$. We assume the matrix elements

$$\langle i|\hat{F}|j\rangle = \begin{pmatrix} 0 & F \\ F & 0 \end{pmatrix}$$

with $F \in \mathbb{R}$. Let the system be at $t = 0$ in the state $|1\rangle$, which means that there is an electron in the ground level.

a) Determine the transition probability $P_{1\to 2}(t)$ as a function of time in lowest order in F using Eq. (5.20) of the compendium.

b) Let $|\Psi^D(t)\rangle = a(t)|1\rangle + b(t)|2\rangle$ hold in the interaction picture. Provide the equations of motion for the amplitudes $a(t)$ and $b(t)$!

c) Solve the equation of motion with the approximation that the rapidly oscillating terms $e^{\pm 2\mathrm{i}\omega t}$ are negligible (which is called *Rotating Wave Approximation*). Compare $|b(t)|^2$ with $P_{1\to 2}(t)$ from part (a).

Exercise 4: *3-dimensional harmonic oscillator in magnetic field*

Aim: Training degenerate perturbation theory

A charged particle with mass m and charge q is confined by a three dimensional harmonic oscillator potential $\hat{H}_0$ with frequency ω. A magnetic field is added which gives rise to a perturbation

$$\hat{H}_p = -\frac{q}{2m} B \hat{L}_z.$$

a) Express $\hat{L}_z$ in step operators.

b) Determine the splitting of the first excited state in first order in B.

Exercise 5: *Rutherford scattering*

Aim: Applying Fermi's golden rule; Getting familiar with a classical experiment

Consider the scattering of α particles at a nucleus with charge Ze (Rutherford Scattering). Here the initial and final states are solutions of the free particle Hamiltonian $\hat{H}_0 = \hat{p}^2/2m_\alpha$, in the form of plane waves $|\mathbf{k}\rangle$. The initial state shall have the wave-vector $\mathbf{k}_i = k_0 \mathbf{e}_z$ and the final state the wave-vector $\mathbf{k}_f$. Scattering is caused by the (screened) Coulomb potential $\phi(r) = Ze \exp(-\lambda r)/(4\pi\epsilon_0 r)$ of the nucleus. Determine the dependence of the transition rate on the angle $\theta = \angle(\mathbf{e}_z, \mathbf{k})$ by Fermi's golden rule in the limit $\lambda \to 0$.

Hint: For the spatial integration, you should choose spherical coordinates with respect to the axis $\mathbf{k}_0 - \mathbf{k}$.

Remark: The result obtained here in lowest order perturbation theory equals both the exact quantum mechanical and the classical result (this is a particular feature of the $1/r$ potential). Comparing experimental data with the classical result, Rutherford thus could deduce in 1911, that the atoms contain a point-like nucleus with positive charge.