LTH Lund
Avd. för Matematisk Fysik
FMFN01/FYSN17

TENTAMENSSKRIVNING KVANTMEKANIK FK

October 26, 2013, 8.00-13.00

Aid: Sheet with formulae, TEFYMA (not needed)

The exam is in total 100 points + 5 bonus points, 7 exercises **(please see also the back of this page!)**

1. *(15 points)* **The one-dimensional harmonic oscillator**

(a) Show that the Hamiltonian

$$\hat{H}_{\mathrm{HO}} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2$$

can be written in the form

$$\hat{H}_{\mathrm{HO}} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$$

and use this expression to calculate the energies of the states $|0\rangle$ and $|1\rangle$.

(b) Derive the first two normalized eigenfunctions of the one-dimensional harmonic oscillator by using step operators, starting from $|0\rangle$.

2. *(15 points)* **Orthonormal complete basis**

Let $\{|u_k\rangle : k = 0, 1, 2, \ldots\}$ be an orthonormal basis that is complete in the Hilbert space.

(a) Express the state $|\Psi\rangle$ in terms of the basis states $|u_k\rangle$.

(b) The observable $\hat{A}$ has eigenstates $|u_k\rangle$ with corresponding eigenvalues λ_k. Assume that the state of the system is

$$|\Psi\rangle = 3\,|u_0\rangle + 4\,|u_1\rangle$$

Now we perform a measurement of $\hat{A}$ on the system. What is the probability $P(\lambda_0), P(\lambda_1), P(\lambda_2)$ of obtaining the result λ_0, λ_1 or λ_2, respectively?

3. *(15 points)* **Commuting observables**

(a) Show that if an operator $\hat{A}$ commutes with two components of an angular momentum operator $\hat{\mathbf{J}}$, then it also commutes with the third component. What follows for $[\hat{A}, \hat{J}^2]$?

(b) If the Hamiltonian $\hat{H}$ commutes with $\hat{L}_z$, what does that tell us about the geometry of the system?

4. *(15 points)* **Angular momentum and spin one**

(a) Consider a system with $j = 1$. Find the matrix representation of $\hat{J}_z, \hat{J}_+$ and $\hat{J}_-$ in the basis

$$|j=1, m_j=1\rangle \leftrightarrow \begin{bmatrix}1\\0\\0\end{bmatrix} \qquad |1,0\rangle \leftrightarrow \begin{bmatrix}0\\1\\0\end{bmatrix} \qquad |1,-1\rangle \leftrightarrow \begin{bmatrix}0\\0\\1\end{bmatrix}$$

(b) An ion in a crystal has spin $s = 1$. For low energies, one only needs to consider the spin degrees of freedom. The Hamiltonian is taken to be

$$\hat{H} = A\hat{S}_z^2 + B(\hat{S}_x^2 - \hat{S}_y^2)$$

where the constants A and B are determined by the crystal lattice structure and the magnetic properties of the surrounding ions. Find the eigenenergies.

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5. *(20 points)* **Magic numbers**

Spin-half particles are moving in a three-dimensional harmonic oscillator potential

$$\hat{V} = \tfrac{1}{2}m\omega_x^2\hat{x}^2 + \tfrac{1}{2}m\omega_y^2\hat{y}^2 + \tfrac{1}{2}m\omega_z^2\hat{z}^2$$

(a) Assume that $\omega_x = \omega_y = \omega_z \equiv \omega$. Determine the four lowest energy levels and their degeneracy. What are the "magic numbers"?

(b) Let $\omega_x = \omega_y = 2\omega_z$. Determine also for this case the four lowest energy levels, their degeneracies and the magic numbers.

(c) The energy levels of the protons or neutrons in some atomic nuclei can be approximated nicely by a spherical harmonic oscillator potential ($\omega_x = \omega_y = \omega_z \equiv \omega$) with corrections of the type $\hat{\mathbf{L}} \cdot \hat{\mathbf{S}}$ and $\hat{L}^2$, i.e. we add

$$\hat{H}_{\text{corr}} = -\kappa\hat{\mathbf{L}} \cdot \hat{\mathbf{S}} - \mu\hat{L}^2$$

to the Hamiltonian, where κ and μ are positive real constants.

(i) Determine to first order the splitting of each energy level due to $\hat{H}_{\text{corr}}$. Here it is best to work in a basis of eigenstates that are common to $\hat{L}^2, \hat{S}^2, \hat{J}^2$ and $\hat{J}_z$: The states $|n_r jl\rangle$, where $n_r = 0, 1, 2, \ldots$ is the radial quantum number, $l = 0, 1, 2, \ldots$ and $j = |l \pm \frac{1}{2}|$, with energies $E_N = \hbar\omega(2n_r + l + \frac{3}{2}) = \hbar\omega(N + \frac{3}{2})$. Note that (quite conveniently) non-degenerate perturbation theory is enough in this case!

(ii) How can the magic number 50 be obtained with this model?

6. *(20 points)* **Level-crossings**

There is a rule in quantum mechanics that says: "Perturbations remove level-crossings". A level-crossing is when the Hamiltonian depends on a parameter and the eigenvalues of it cross if plotted as a function of this parameter. We shall investigate this in a very simple model. Consider a two dimensional state space where the unperturbed Hamiltonian $\hat{H}_0$ and the perturbation $\hat{H}_p$ are given by the matrices:

$$H_0(\gamma) = \begin{bmatrix} k\gamma & 0 \\ 0 & -k\gamma \end{bmatrix}, \qquad H_p = \begin{bmatrix} 0 & \varepsilon \\ \varepsilon^* & 0 \end{bmatrix}$$

Here, k is a positive real constant, ε is a constant, and γ is a parameter $(-1 < \gamma < 1)$.

(a) Plot the eigenvalues of $\hat{H}_0$ as a function of γ.

(b) Calculate the eigenvalues of $\hat{H}_0 + \hat{H}_p$ and plot those in the same diagram.

(c) What is the result if we use perturbation theory (non-degenerate is sufficient)? Comments?

7. *(5 bonus points)* **Contact-interacting fermions**

Six fermions are in a one-dimensional harmonic oscillator potential. They interact with a contact interaction, which is given by

$$V_{\text{int}}(x_1, x_2) = g\delta(x_1 - x_2)$$

where g is a real constant. All the fermions have spin up. Find the three lowest energy levels of the system.

Good luck!

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TENTAMENSSKRIVNING KVANTMEKANIK FK

October 19, 2010, 14.00-19.00

Aid: Sheet with formulae, pocket calculator (not needed), TEFYMA (not needed)

The exam is in total 100 points + 5 bonus points.

Operators are denoted by boldface letters (e.g. $\boldsymbol{H}$) and vectors with an arrow (e.g. $\vec{x}$). You can choose whatever notation you like best, just make it clear what notation you use and stick to it.

1. *(4 points)* Consider two kets $|\Psi\rangle$ and $|\Psi'\rangle$ such that $|\Psi'\rangle = \mathrm{e}^{\mathrm{i}\theta}|\Psi\rangle$, where θ is a real number.

(a) Prove that if $|\Psi\rangle$ is normalized, so is $|\Psi'\rangle$.
(b) Demonstrate that the predicted probabilities for an arbitrary measurement are the same for $|\Psi\rangle$ and $|\Psi'\rangle$; therefore, $|\Psi\rangle$ and $|\Psi'\rangle$ represent the same physical state.

2. *(10 points)* Consider a harmonic oscillator of mass M and angular frequency ω. Its state is given in terms of its eigenstates

$$|\Psi\rangle = \sum_{n=0}^{\infty} c_n |n\rangle, \qquad c_n \in \mathbb{C}$$

(a) What is the probability $\mathcal{P}$ that a measurement of the oscillator's energy will yield a result greater than $2\hbar\omega$? When $\mathcal{P} = 0$, what are the non-zero coefficients c_n?
(b) Now, assume that only c_0 and c_1 are different from zero. Write the normalization condition for $|\Psi\rangle$ and the mean value $\langle \boldsymbol{H} \rangle$ of the energy in terms of c_0 and c_1.
(c) If we also require $\langle \boldsymbol{H} \rangle = \hbar\omega$, calculate $|c_0|^2$ and $|c_1|^2$.

3. *(20 points)* There is a rule in quantum mechanics that says: "Perturbations remove level-crossings." A level-crossing is when the Hamiltonian depends on a parameter and the eigenvalues as a function of this parameter cross. We shall investigate this in a very simple model. Consider a two-dimensional state space where the unperturbed Hamiltonian $\boldsymbol{H}_0$ and the perturbation $\boldsymbol{H}_p$ are given by

$$\boldsymbol{H}_0(\lambda) = \begin{bmatrix} k\lambda & 0 \\ 0 & -k\lambda \end{bmatrix}, \qquad \boldsymbol{H}_p = \begin{bmatrix} 0 & c \\ c^* & 0 \end{bmatrix}$$

Here k is a positive real constant, c is a complex constant and λ a parameter ($-1 < \lambda < 1$).

(a) Find the eigenvalues of $\boldsymbol{H}_0$. Sketch them as a function of λ.
(b) Calculate the eigenvalues of $\boldsymbol{H}_0 + \boldsymbol{H}_p$ and sketch those in the same diagram.
(c) What is the result if we use perturbation theory? Comments?

4. *(8 points)*

(a) Prove that if an orthonormal discrete set of kets $\{|u_i\rangle,\ i = 1, 2, \ldots\}$ constitutes a basis, then it follows that

$$\sum_i |u_i\rangle \langle u_i| = \mathbb{1}$$

(b) Show that if the closure relation is valid for an orthonormal continuous set $\{|w_\alpha\rangle\}$, then this set constitutes a basis.

5. *(30 points)* A two-dimensional harmonic oscillator of mass M and angular frequency ω is perturbed by

$$\boldsymbol{H}_p = \lambda M\omega^2 \boldsymbol{xy}$$

(a) Find the first order correction to the ground state energy.

(b) Find the second order correction to the ground state energy.

(c) Solve the full problem exactly and compare the result with the approximation you obtained in parts (a) and (b).
Hint: Use the variable substitution $x' = \frac{1}{\sqrt{2}}(x+y)$, $y' = \frac{1}{\sqrt{2}}(x-y)$.

6. *(8 points)* In the system of a boson with spin $s = 1$, build the matrices of S_x, S_y and S_z in a basis of the eigenstates of S^2 and S_z. Check that $[S_x, S_y] = i\hbar S_z$ holds.

7. *(20 points)* A system has the wave function

$$\Psi(\vec{r}) = N(z-x)e^{-r/\alpha}, \qquad r = \sqrt{x^2+y^2+z^2}$$

where α is real and N is a normalization constant.

(a) Writing $\Psi(\vec{r})$ on the form $\Psi(\vec{r}) = R(r)F(\theta,\varphi)$, normalize the angular part $F(\theta,\varphi)$.
Recall that:

$$Y_1^0(\theta,\varphi) = \sqrt{\frac{3}{4\pi}}\cos\theta, \qquad Y_1^{\pm1}(\theta,\varphi) = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\varphi}$$

(b) The observables L_z and L^2 are measured. What are their possible outcomes and what are their probabilities?

(c) *(5 bonus points)* What are the possible outcomes of a measurement of L_x?
(Note that with a simple observation, the previous procedure can be applied.)

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TENTAMENSSKRIVNING KVANTMEKANIK FK

December 16, 2010, 08.00-13.00

Aid: Sheet with formulae, TEFYMA (not needed)

The exam is in total 100 points + 5 bonus points, 6 exercises

Operators are denoted by boldface letters (e.g. $\boldsymbol{H}$) and vectors with an arrow (e.g. $\vec{x}$). You can choose whatever notation you like best, just make it clear what notation you use and stick to it.

1. *(6 points)* Let $|\psi_1\rangle$ and $|\psi_2\rangle$ be two orthogonal normalized states of a physical system:

$$\langle\psi_1|\psi_2\rangle = 0 \qquad \text{and} \qquad \langle\psi_1|\psi_1\rangle = \langle\psi_2|\psi_2\rangle = 1$$

and let $\boldsymbol{A}$ be an observable of the system. Consider a nondegenerate eigenvalue of $\boldsymbol{A}$ denoted by α_n to which the normalized state $|\phi_n\rangle$ corresponds. We define

$$P_1(\alpha_n) = |\langle\phi_n|\psi_1\rangle|^2 \qquad \text{and} \qquad P_2(\alpha_n) = |\langle\phi_n|\psi_2\rangle|^2$$

(a) What is the interpretation of $P_1(\alpha_n)$ and $P_2(\alpha_n)$?
(b) A given particle is in the state $|\Phi\rangle = 3\,|\psi_1\rangle - 4i\,|\psi_2\rangle$. Normalize $|\Phi\rangle$.
(c) Now assume that $\langle\phi_n|\psi_1\rangle, \langle\phi_n|\psi_2\rangle$ are real numbers and the system is in the state $|\Phi\rangle$. What is the probability of getting α_n when $\boldsymbol{A}$ is measured? Write the answer in terms of $P_1(\alpha_n)$ and $P_2(\alpha_n)$.

Solution.

(a) According to the postulates of quantum mechanics, $P_1(\alpha_n)$ is the probability of obtaining α_n when $\boldsymbol{A}$ is measured, while the system is in the state $|\psi_1\rangle$. The same is true for $P_2(\alpha_n)$ in the state $|\psi_2\rangle$.
(b) The norm of $|\Phi\rangle$ is

$$\begin{aligned}\sqrt{\langle\Phi|\Phi\rangle} &= \sqrt{(3\,\langle\psi_1| + 4\mathrm{i}\,\langle\psi_2|)(3\,|\psi_1\rangle - 4\mathrm{i}\,|\psi_2\rangle)} \\ &= \sqrt{9\langle\psi_1|\psi_1\rangle + 16\langle\psi_2|\psi_2\rangle + 12\mathrm{i}\langle\psi_2|\psi_1\rangle - 12\mathrm{i}\langle\psi_1|\psi_2\rangle} \\ &= \sqrt{9+16+0+0} = \sqrt{25} = 5\end{aligned}$$

Hence, the normalized state is

$$|\Phi'\rangle = \tfrac{1}{5}\,(3\,|\psi_1\rangle - 4\mathrm{i}\,|\psi_2\rangle)$$

(c) The probability of getting α_n when measuring $\boldsymbol{A}$ is

$$\begin{aligned}P(\alpha_n) &= |\langle\phi_n|\Phi'\rangle|^2 = \left|\langle\phi_n|\left[\tfrac{1}{5}\,(3\,|\psi_1\rangle - 4\mathrm{i}\,|\psi_2\rangle)\right]\right|^2 \\ &= \tfrac{1}{25}\,|3\langle\phi_n|\psi_1\rangle - 4\mathrm{i}\langle\phi_n|\psi_2\rangle|^2 \\ &= \tfrac{1}{25}\,[3\langle\phi_n|\psi_1\rangle - 4\mathrm{i}\langle\phi_n|\psi_2\rangle)^*\,(3\langle\phi_n|\psi_1\rangle - 4\mathrm{i}\langle\phi_n|\psi_2\rangle] \\ &= \tfrac{1}{25}[3\langle\phi_n|\psi_1\rangle^* + 4\mathrm{i}\langle\phi_n|\psi_2\rangle^*)(3\langle\phi_n|\psi_1\rangle - 4\mathrm{i}\langle\phi_n|\psi_2\rangle] \\ &= \tfrac{1}{25}[9|\langle\phi_n|\psi_1\rangle|^2 + 16|\langle\phi_n|\psi_2\rangle|^2 + 12\mathrm{i}\langle\phi_n|\psi_1\rangle\langle\phi_n|\psi_2\rangle^* - 12\mathrm{i}\langle\phi_n|\psi_1\rangle^*\langle\phi_n|\psi_2\rangle] \\ &= \tfrac{1}{25}\,[9P_1(\alpha_n) + 16P_2(\alpha_n)]\end{aligned}$$

2. *(6 points)*

(a) Let $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ be operators. Prove the commutator relation

$$[\boldsymbol{A}, \boldsymbol{BC}] = [\boldsymbol{A}, \boldsymbol{B}]\boldsymbol{C} + \boldsymbol{B}[\boldsymbol{A}, \boldsymbol{C}]$$

(b) Calculate the commutator $[\boldsymbol{p}_x^2, \boldsymbol{x}^2]$.

(c) Let $f(x)$ be a differentiable function. Compute $[\boldsymbol{p}_x, f(\boldsymbol{x})]$.

Solution.

(a)
$$\begin{aligned}[\boldsymbol{A}, \boldsymbol{B}]\boldsymbol{C} + \boldsymbol{B}[\boldsymbol{A}, \boldsymbol{C}] &= (\boldsymbol{AB} - \boldsymbol{BA})\boldsymbol{C} + \boldsymbol{B}(\boldsymbol{AC} - \boldsymbol{CA}) = \boldsymbol{ABC} - \boldsymbol{BAC} + \boldsymbol{BAC} - \boldsymbol{BCA} \\ &= \boldsymbol{ABC} - \boldsymbol{BCA} = [\boldsymbol{A}, \boldsymbol{BC}]\end{aligned}$$

(b) We use the commutator relation from (a) and remember that $[\boldsymbol{p}, \boldsymbol{x}] = -\mathrm{i}\hbar$, giving

$$\begin{aligned}[\boldsymbol{p}^2, \boldsymbol{x}^2] &= \boldsymbol{x}[\boldsymbol{p}^2, \boldsymbol{x}] + [\boldsymbol{p}^2, \boldsymbol{x}]\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{p}[\boldsymbol{p}, \boldsymbol{x}] + [\boldsymbol{p}, \boldsymbol{x}]\boldsymbol{p}) + (\boldsymbol{p}[\boldsymbol{p}, \boldsymbol{x}] + [\boldsymbol{p}, \boldsymbol{x}]\boldsymbol{p})\boldsymbol{x} = -2\mathrm{i}\hbar(\boldsymbol{xp} + \boldsymbol{px}) \\ &= -2\mathrm{i}\hbar(\boldsymbol{xp} + (-\mathrm{i}\hbar + \boldsymbol{xp})) = -2\hbar^2 - 4\mathrm{i}\hbar\boldsymbol{xp}\,.\end{aligned}$$

(c) Let $g(x)$ be some well-behaved test function. Then

$$\begin{aligned}[\boldsymbol{p}, f(\boldsymbol{x})]g(x) &= (\boldsymbol{p}f(\boldsymbol{x}) - f(\boldsymbol{x})\boldsymbol{p})g(x) = \boldsymbol{p}f(\boldsymbol{x})g(x) - f(\boldsymbol{x})\boldsymbol{p}g(x) = -\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}x}(f(x)g(x)) + \mathrm{i}\hbar f(x)\frac{\mathrm{d}}{\mathrm{d}x}g(x) \\ &= -\mathrm{i}\hbar\left(\frac{\mathrm{d}f}{\mathrm{d}x}g(x) + f\frac{\mathrm{d}g}{\mathrm{d}x}\right) + \mathrm{i}\hbar f(x)\frac{\mathrm{d}g}{\mathrm{d}x} = -\mathrm{i}\hbar\frac{\mathrm{d}f}{\mathrm{d}x}g\end{aligned}$$

and therefore

$$[\boldsymbol{p}, f(\boldsymbol{x})] = -\mathrm{i}\hbar f'(\boldsymbol{x})\,. \qquad \square$$

3. *(30 points)* Consider a one-dimensional harmonic oscillator that is perturbed by

$$\boldsymbol{H}_p = \alpha\hbar\omega\left(\frac{2m\omega}{\hbar}\right)^{3/2}\boldsymbol{x}^3$$

where α is a real constant.

(a) Express $\boldsymbol{H}_p$ in the operators $\boldsymbol{a}^\dagger$ and $\boldsymbol{a}$.

(b) Determine the matrix elements $\langle m|\boldsymbol{H}_p|n\rangle$, where $|n\rangle \equiv |\phi_n\rangle$ is the nth eigenstate of the unperturbed 1D harmonic oscillator.

(c) Consider the second excited state. Calculate the 1st and 2nd order correction to the energy, and the 1st order correction to the wave function.

Solution.

(a) From the formula sheet we know that

$$\boldsymbol{x} = \frac{1}{2}\sqrt{\frac{2\hbar}{m\omega}}(\boldsymbol{a}^\dagger + \boldsymbol{a})$$

which gives (using $\boldsymbol{aa}^\dagger - \boldsymbol{a}^\dagger\boldsymbol{a} = 1$ repeatedly)

$$\begin{aligned}\boldsymbol{H}_p &= \alpha\hbar\omega\left(\frac{2m\omega}{\hbar}\right)^{3/2}\boldsymbol{x}^3 = \alpha\hbar\omega\left(\frac{2m\omega}{\hbar}\right)^{3/2}\frac{1}{8}\left(\frac{2\hbar}{m\omega}\right)^{3/2}(\boldsymbol{a}^\dagger + \boldsymbol{a})^3 \\ &= \alpha\hbar\omega\left((\boldsymbol{a}^\dagger)^3 + (\boldsymbol{a}^\dagger)^2\boldsymbol{a} + \boldsymbol{a}^\dagger\boldsymbol{a}\boldsymbol{a}^\dagger + \boldsymbol{a}(\boldsymbol{a}^\dagger)^2 + \boldsymbol{a}^\dagger\boldsymbol{a}^2 + \boldsymbol{a}\boldsymbol{a}^\dagger\boldsymbol{a} + \boldsymbol{a}^2\boldsymbol{a}^\dagger + \boldsymbol{a}^3\right) \\ &= \alpha\hbar\omega\left((\boldsymbol{a}^\dagger)^3 + (\boldsymbol{a}^\dagger)^2\boldsymbol{a} + \boldsymbol{a}^\dagger(\boldsymbol{a}^\dagger\boldsymbol{a} + 1) + (\boldsymbol{a}^\dagger\boldsymbol{a} + 1)\boldsymbol{a}^\dagger + \boldsymbol{a}^\dagger\boldsymbol{a}^2 + (\boldsymbol{a}^\dagger\boldsymbol{a} + 1)\boldsymbol{a} + \boldsymbol{a}(\boldsymbol{a}^\dagger\boldsymbol{a} + 1) + \boldsymbol{a}^3\right) \\ &= \alpha\hbar\omega\left((\boldsymbol{a}^\dagger)^3 + 3(\boldsymbol{a}^\dagger)^2\boldsymbol{a} + \boldsymbol{a}^\dagger + \boldsymbol{a}^\dagger(\boldsymbol{a}^\dagger\boldsymbol{a} + 1) + \boldsymbol{a}^\dagger + 2\boldsymbol{a}^\dagger\boldsymbol{a}^2 + \boldsymbol{a} + (\boldsymbol{a}^\dagger\boldsymbol{a} + 1)\boldsymbol{a} + \boldsymbol{a} + \boldsymbol{a}^3\right) \\ &= \alpha\hbar\omega\left((\boldsymbol{a}^\dagger)^3 + 3(\boldsymbol{a}^\dagger)^2\boldsymbol{a} + 3\boldsymbol{a}^\dagger + 3\boldsymbol{a}^\dagger\boldsymbol{a}^2 + 3\boldsymbol{a} + \boldsymbol{a}^3\right) \\ &= \alpha\hbar\omega\left((\boldsymbol{a}^\dagger)^3 + 3\boldsymbol{a}^\dagger(\boldsymbol{a}^\dagger\boldsymbol{a} + 1) + 3(\boldsymbol{a}^\dagger\boldsymbol{a} + 1)\boldsymbol{a} + \boldsymbol{a}^3\right)\end{aligned}$$

(b) Using the result of (a)

$$\begin{aligned}
\langle m|\boldsymbol{H}_p|n\rangle &= \alpha\hbar\omega\langle m|\left((\boldsymbol{a}^\dagger)^3 + 3\boldsymbol{a}^\dagger(\boldsymbol{a}^\dagger\boldsymbol{a}+1) + 3(\boldsymbol{a}^\dagger\boldsymbol{a}+1)\boldsymbol{a} + \boldsymbol{a}^3\right)|n\rangle \\
&= \alpha\hbar\omega\,\langle m|\left((\boldsymbol{a}^\dagger)^3|n\rangle + 3\boldsymbol{a}^\dagger(\boldsymbol{a}^\dagger\boldsymbol{a}+1)|n\rangle + 3(\boldsymbol{a}^\dagger\boldsymbol{a}+1)\boldsymbol{a}|n\rangle + \boldsymbol{a}^3|n\rangle\right) \\
&= \alpha\hbar\omega\,\langle m|\Big(\sqrt{(n+1)(n+2)(n+3)}\,|n+3\rangle + 3\sqrt{n+1}(n+1)\,|n+1\rangle \\
&\qquad +3((n-1)+1)\sqrt{n}\,|n-1\rangle + \sqrt{n(n-1)(n-2)}\,|n-3\rangle\Big) \\
&= \alpha\hbar\omega\Big(\sqrt{(n+1)(n+2)(n+3)}\delta_{m,n+3} + 3(n+1)^{3/2}\delta_{m,n+1} \\
&\qquad +3n^{3/2}\delta_{m,n-1} + \sqrt{n(n-1)(n-2)}\delta_{m,n-3}\Big)
\end{aligned}$$

(c) For the 2nd excited state we have $n=2$. From (b) we know that $\Delta E_2^{(1)} = \langle 2|\boldsymbol{H}_p|2\rangle = 0$, that is the 1st order correction to the energy is zero. For the 2nd order correction we find

$$\begin{aligned}
\Delta E_2^{(2)} &= \sum_{m\neq 2}\frac{|\langle m|\boldsymbol{H}_p|2\rangle|^2}{E_2^{(0)} - E_m^{(0)}} \\
&= \left[\frac{|\alpha\hbar\omega\sqrt{3\cdot 4\cdot 5}|^2}{\hbar\omega\left(2+\frac{1}{2}\right) - \hbar\omega\left(5+\frac{1}{2}\right)} + \frac{|\alpha\hbar\omega 3\cdot 3^{3/2}|^2}{\hbar\omega\left(2+\frac{1}{2}\right) - \hbar\omega\left(3+\frac{1}{2}\right)} + \frac{|\alpha\hbar\omega 3\cdot 2^{3/2}|^2}{\hbar\omega\left(2+\frac{1}{2}\right) - \hbar\omega\left(1+\frac{1}{2}\right)} + 0\right] \\
&= -191\alpha^2\hbar\omega
\end{aligned}$$

The 1st order correction to the wave function is

$$\begin{aligned}
\Delta\psi_2^{(1)} &= \sum_{m\neq 2}\frac{\langle m|\boldsymbol{H}_p|2\rangle}{E_2^{(0)} - E_m^{(0)}}\phi_m \\
&= \left[\frac{\alpha\hbar\omega\sqrt{3\cdot 4\cdot 5}}{\hbar\omega\left(2+\frac{1}{2}\right) - \hbar\omega\left(5+\frac{1}{2}\right)}\phi_5 + \frac{\alpha\hbar\omega 3(3)^{3/2}}{\hbar\omega\left(2+\frac{1}{2}\right) - \hbar\omega\left(3+\frac{1}{2}\right)}\phi_3 + \frac{\alpha\hbar\omega 3(2)^{3/2}}{\hbar\omega\left(2+\frac{1}{2}\right) - \hbar\omega\left(1+\frac{1}{2}\right)}\phi_1\right] \\
&= \alpha\left(6\sqrt{2}\phi_1 - 9\sqrt{3}\phi_3 - \sqrt{\frac{20}{3}}\phi_5\right)
\end{aligned}$$

4. *(30 points)* A two-dimensional harmonic oscillator of mass M and angular frequency ω is perturbed by

$$\boldsymbol{H}_p = \lambda M\omega^2\boldsymbol{x}\boldsymbol{y}$$

(a) Find the first order correction to the ground state energy.
(b) Find the second order correction to the ground state energy.
(c) Solve the full problem exactly and compare the result with the approximation you obtained in parts (a) and (b).
Hint: Use the variable substitution $x' = \frac{1}{\sqrt{2}}(x+y)$, $y' = \frac{1}{\sqrt{2}}(x-y)$.

Solution. For energy levels, wave functions, etc. of the unperturbed 2-dimensional harmonic oscillator, refer to p. 19 in the compendium.

(a) For the ground state we have $n=0$ and hence $n_x = n_y = 0$. Note that the eigenfunctions of the two-dimensional harmonic oscillator are $\phi_{n_xn_y}(x,y) = \phi_{n_x}(x)\phi_{n_y}(y)$ where $\phi_{n_x}(x)$ $[\phi_{n_y}(y)]$ is the n_xth $[n_y$th$]$ eigenfunction of the *one*-dimensional harmonic oscillator. Now

$$\begin{aligned}
\Delta E_0^{(1)} = \langle 00|H_p|00\rangle &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(\phi_0(x)\phi_0(y))^*xy\phi_0(x)\phi_0(y)\mathrm{d}x\mathrm{d}y \\
&= \int_{-\infty}^{+\infty}\underbrace{x|\phi_0(x)|^2}_{\text{odd function}}\,\mathrm{d}x\int_{-\infty}^{+\infty}\underbrace{y|\phi_0(y)|^2}_{\text{odd function}}\,\mathrm{d}y = 0
\end{aligned}$$

Another way is to employ step operators (here I use the notation $|n_x n_y\rangle = |n_x\rangle_x |n_y\rangle_y$ and note that the x-operators only work on the x-ket, y-operators only work on the y-ket)

$$\begin{aligned}\Delta E_0^{(1)} &= \langle 00|H_p|00\rangle \propto \langle 00|(a_x^\dagger + a_x)(a_y^\dagger + a_y)|00\rangle = \langle 0|_x \langle 0|_y \left(a_x^\dagger a_y^\dagger + a_x^\dagger a_y + a_x a_y^\dagger + a_x a_y\right)|0\rangle_x |0\rangle_y \\ &= \langle 0|_x \langle 0|_y \Big(|1\rangle_x |1\rangle_y + 0 + 0 + 0\Big) = 0\end{aligned}$$

(b) For the 2nd order correction to the ground state ($n_x = n_y = 0$) we need

$$\begin{aligned}\langle n_x n_y|H_p|00\rangle &= \tfrac{1}{2}\lambda\hbar\omega\langle n_x n_y|\left(a_x^\dagger a_y^\dagger + a_x^\dagger a_y + a_x a_y^\dagger + a_x a_y\right)|00\rangle = \tfrac{1}{2}\lambda\hbar\omega\,\langle n_x n_y|\left(|11\rangle + 0 + 0 + 0\right) \\ &= \tfrac{1}{2}\lambda\hbar\omega\delta_{n_x,1}\delta_{n_y,1}\end{aligned}$$

which gives

$$\Delta E_0^{(2)} = \sum_{(n_x,n_y)\neq(0,0)} \frac{|\langle n_x n_y|H_p|00\rangle|^2}{E_0^{(0)} - E_{n_x+n_y}^{(0)}} = \frac{|\frac{1}{2}\lambda\hbar\omega|^2}{\hbar\omega(0+1) - \hbar\omega((1+1)+1)} = -\tfrac{1}{8}\lambda^2\hbar\omega$$

(c) The full Hamiltonian is

$$H = \frac{p^2}{2m} + \tfrac{1}{2}m\omega^2\left(x^2+y^2\right) + \lambda m\omega^2 xy = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \tfrac{1}{2}m\omega^2\left(x^2+y^2+2\lambda xy\right)$$

We would like to be able to put it on a harmonic oscillator form, i.e. only with 2nd order terms in some coordinates (x', y'). Note that if

$$x \propto x' + y' \text{ and } y \propto x' - y' \qquad \text{then} \qquad xy \propto x'^2 - y'^2 \text{ and } x^2 + y^2 \propto x'^2 + y'^2$$

Now one would substitute (x, y) by $(x'+y', x'-y')$, find out what constant to put in front of $x'+y'$ and $x'-y'$ and put that in afterwards. But I've already gone through the calculations so I'll use the most convenient constant directly, i.e.

$$x = \frac{1}{\sqrt{2}}(x'+y'), \qquad y = \frac{1}{\sqrt{2}}(x'-y')$$

which ensures that $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2}$. Now,

$$x' = \frac{1}{\sqrt{2}}(x+y), \qquad y' = \frac{1}{\sqrt{2}}(x-y)$$

and we find

$$\begin{aligned}\frac{\partial}{\partial x} &= \frac{\partial}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial}{\partial y'}\frac{\partial y'}{\partial x} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x'} + \frac{\partial}{\partial y'}\right) \\ \frac{\partial}{\partial y} &= \frac{\partial}{\partial x'}\frac{\partial x'}{\partial y} + \frac{\partial}{\partial y'}\frac{\partial y'}{\partial y} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x'} - \frac{\partial}{\partial y'}\right)\end{aligned}$$

$$\left.\begin{aligned}\frac{\partial^2}{\partial x^2} &= \frac{1}{2}\left(\frac{\partial}{\partial x'} + \frac{\partial}{\partial y'}\right)^2 = \frac{1}{2}\left(\frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} + 2\frac{\partial}{\partial x'}\frac{\partial}{\partial y'}\right) \\ \frac{\partial^2}{\partial y^2} &= \frac{1}{2}\left(\frac{\partial}{\partial x'} - \frac{\partial}{\partial y'}\right)^2 = \frac{1}{2}\left(\frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} - 2\frac{\partial}{\partial x'}\frac{\partial}{\partial y'}\right)\end{aligned}\right\} \quad \Rightarrow \quad \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2}$$

Finally we substitute the variables

$$\begin{aligned}H &= -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \tfrac{1}{2}m\omega^2\left(x^2+y^2+2\lambda xy\right) \\ &= -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2}\right) + \tfrac{1}{2}m\omega^2\left(x'^2+y'^2+2\lambda\tfrac{1}{2}\left(x'^2-y'^2\right)\right) \\ &= \frac{p'^2}{2m} + \tfrac{1}{2}m\left((\omega\sqrt{1+\lambda})^2 x'^2 + (\omega\sqrt{1-\lambda})^2 y'^2\right) \\ &\equiv \frac{p'^2}{2m} + \tfrac{1}{2}m\left(\omega_{x'}^2 x'^2 + \omega_{y'}^2 y'^2\right)\end{aligned}$$

This is the Hamiltonian for an anisotropic ($\omega_{x'} \neq \omega_{y'}$) harmonic oscillator in two dimensions. Its eigenenergies are

$$E_{n_{x'}n_{y'}} = \hbar\omega_{x'}(n_{x'} + \tfrac{1}{2}) + \hbar\omega_{x'}(n_{y'} + \tfrac{1}{2}) = \hbar\omega\left(\sqrt{1+\lambda}(n_{x'} + \tfrac{1}{2}) + \sqrt{1-\lambda}(n_{y'} + \tfrac{1}{2})\right)$$

Assuming that λ is small, the exact energy can be Taylor expanded to 2nd order

$$\begin{aligned}
E_{n_{x'}n_{y'}} &\approx \hbar\omega\left(\left(1 + \tfrac{1}{2}\lambda - \tfrac{1}{8}\lambda^2\right)(n_{x'} + \tfrac{1}{2}) + \left(1 - \tfrac{1}{2}\lambda - \tfrac{1}{8}\lambda^2\right)(n_{y'} + \tfrac{1}{2})\right) \\
&= \hbar\omega(n_{x'} + n_{y'} + 1) + \tfrac{1}{2}\lambda\hbar\omega(n_{x'} - n_{y'}) - \tfrac{1}{8}\lambda^2\hbar\omega(n_{x'} + n_{y'} + 1) \\
&= \underbrace{\hbar\omega(n+1)}_{=E_n^{(0)}} + \underbrace{\tfrac{1}{2}\lambda\hbar\omega(n_{x'} - n_{y'})}_{=\Delta E_n^{(1)}} + \underbrace{(-\tfrac{1}{8})\lambda^2\hbar\omega(n+1)}_{=\Delta E_n^{(2)}}
\end{aligned}$$

Note how the 1st and 2nd order corrections we found in (a) and (b) appear in the Taylor expansion.

5. *(8 points)*

(a) Construct the matrix for the operator $\boldsymbol{S}_+$ in the space $\mathcal{M}^{1/2}$.
(b) Show by using matrix multiplication that $\boldsymbol{S}_+\alpha = 0$.
(c) Determine by matrix multiplication the constant c in the relation

$$\boldsymbol{S}_+\beta = c\alpha$$

and compare the result with the expression for general angular momentum.

Solution. In the space $\mathcal{M}^{1/2}$ we have $s = \frac{1}{2}$ and therefore $m_s \in \{\frac{1}{2}, -\frac{1}{2}\}$, so the space is 2-dimensional with basis $\{|\frac{1}{2}\rangle, |-\frac{1}{2}\rangle\}$, whose matrix representation is $\left\{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\right\}$.

(a) We have

$$\langle m_s'|\,\boldsymbol{S}_+\,|m_s\rangle = \hbar\sqrt{(\tfrac{1}{2} - m_s)(\tfrac{1}{2} + m_s + 1)}\langle m_s'|m_s + 1\rangle = \hbar\sqrt{(\tfrac{1}{2} - m_s)(\tfrac{1}{2} + m_s + 1)}\delta_{m_s', m_s+1}$$

giving

$$\boldsymbol{S}_+ = \begin{bmatrix} 0 & \hbar \\ 0 & 0 \end{bmatrix}$$

(b) The state α corresponds to spin up, that is $|\frac{1}{2}\rangle$, with matrix representation $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$. Hence

$$\boldsymbol{S}_+\alpha = \begin{bmatrix} 0 & \hbar \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(c) The state β corresponds to spin down, that is $|-\frac{1}{2}\rangle$, with matrix representation $\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$. Thus

$$\boldsymbol{S}_+\beta = \begin{bmatrix} 0 & \hbar \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \hbar \\ 0 \end{bmatrix} = \hbar\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \hbar\alpha$$

so the constant is $c = \hbar$. The general expression is

$$c = \hbar\sqrt{(s - m_s)(s + m_s + 1)} = \hbar\sqrt{(\tfrac{1}{2} - (-\tfrac{1}{2})(\tfrac{1}{2} + (-\tfrac{1}{2}) + 1)} = \hbar$$

6. *(20 points)* At a certain moment the wave function for a particle is given by

$$\phi(\vec{r}) = f(r)U(\theta,\varphi)$$

where

$$U(\theta,\varphi) = N(\sin\theta\cos\varphi + \cos\theta + 1)$$

and N is a normalization constant.

(a) Write $U(\theta,\varphi)$ in terms of the spherical harmonics $Y_l^m(\theta,\varphi)$. The first few of them are:

$$Y_0^0(\theta,\varphi) = \sqrt{\frac{1}{4\pi}} \qquad Y_2^0(\theta,\varphi) = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1)$$

$$Y_1^0(\theta,\varphi) = \sqrt{\frac{3}{4\pi}}\cos\theta \qquad Y_2^{\pm1}(\theta,\varphi) = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{\pm i\varphi}$$

$$Y_1^{\pm1}(\theta,\varphi) = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\varphi} \qquad Y_2^{\pm2}(\theta,\varphi) = \sqrt{\frac{15}{32\pi}}\sin^2\theta e^{\pm 2i\varphi}$$

(b) Determine N such that $U(\theta,\varphi)$ is normalized.

(c) The observables $\boldsymbol{L}_z$ and $\boldsymbol{L}^2$ are measured. What are their possible outcomes and what are their probabilities?

(d) *(5 bonus points)* What are the possible outcomes of a measurement of $\boldsymbol{L}_x$?
(Note that by a little use of cartesian coordinates and with a simple observation, the previous procedure can be applied.)

Solution.

(a)

$$U(\theta,\varphi) = N\left(\sin\theta\tfrac{1}{2}\left(e^{i\varphi} + e^{-i\varphi}\right) + \cos\theta + 1\right) = N\left(\frac{1}{2}\sqrt{\frac{8\pi}{3}}\left(Y_1^{-1} - Y_1^1\right) + \sqrt{\frac{4\pi}{3}}Y_1^0 + \sqrt{4\pi}Y_0^0\right)$$

(b) The spherical harmonics Y_l^m are orthonormal, so we find right away

$$1 = \langle U|U\rangle = N^2\left(2\frac{2\pi}{3} + \frac{4\pi}{3} + 4\pi\right) = N^2\frac{20\pi}{3} \quad \Rightarrow \quad N = \sqrt{\frac{3}{5}}\frac{1}{\sqrt{4\pi}}$$

giving

$$U(\theta,\varphi) = \frac{1}{\sqrt{10}}\left(Y_1^{-1} - Y_1^1\right) + \frac{1}{\sqrt{5}}Y_1^0 + \sqrt{\frac{3}{5}}Y_0^0$$

(c) From the result in (b) we see that the possible values of l are $l = 0, 1$ and the possible values of m are $m = -1, 0, 1$. Hence $\boldsymbol{L}^2$ can be measured to $l(l+1)\hbar^2 = 0, 2\hbar^2$ with probabilities $P(l=0) = \frac{3}{5}$ and $P(l=1) = 2\frac{1}{10} + \frac{1}{5} = \frac{2}{5}$, and $\boldsymbol{L}_z$ can be measured to $m\hbar = -h, 0, h$ with probabilities $P(m=-1) = \frac{1}{10}$, $P(m=0) = \frac{1}{5} + \frac{3}{5} = \frac{4}{5}$ and $P(m=1) = \frac{1}{10}$.

(d) Note that

$$\phi(\vec{r}) = f(r)U(\theta,\phi) = Nf(r)\frac{1}{r}(r\sin\theta\cos\varphi + r\cos\theta + 1) = Nf(r)\frac{1}{r}(x + z + r)$$

The wave function ϕ is therefore symmetric in x and z. Therefore the possible values when measuring $\boldsymbol{L}_x$ and their probabilities are the same as for $\boldsymbol{L}_z$. That is, we can measure $\boldsymbol{L}_x$ to $-\hbar, 0, \hbar$ with the respective probabilities $\frac{1}{10}, \frac{4}{5}, \frac{1}{10}$.